Existence and Nonlinear Stability of Rotating Star Solutions of the Compressible Euler-Poisson Equations

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Abstract

We prove existence of rotating star solutions which are steady-state solutions of the compressible isentropic Euler-Poisson (EP) equations in 3 spatial dimensions, with prescribed angular momentum and total mass. This problem can be formulated as a variational problem of finding a minimizer of an energy functional in a broader class of functions having less symmetry than those functions considered in the classical Auchmuty-Beals paper. We prove the nonlinear dynamical stability of these solutions with perturbations having the same total mass and symmetry as the rotating star solution. We also prove local in time stability of $W^{1,\infty}(\mathbb{R}^3)$ solutions where the perturbations are entropy-weak solutions of the EP equations. Finally, we give a uniform (in time) a-priori estimate for entropy-weak solutions of the EP equations.

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1 Introduction

The motion of a compressible isentropic perfect fluid with self-gravitation is modeled by the Euler-Poisson equations in three space dimensions (cf [4]):

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p(\rho) = -\rho \nabla \Phi, \\ \Delta \Phi = 4\pi \rho. \end{cases} \tag{1.1}$$

Here ρ, $\mathbf{v} = (v_1, v_2, v_3)$, $p(\rho)$ and Φ denote the density, velocity, pressure and gravitational potential, respectively. The gravitational potential is given by

$$\Phi(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy = -\rho * \frac{1}{|x|}, \tag{1.2}$$

where $*$ denotes convolution. The momentum $\rho \mathbf{v}$ is denoted by $\mathbf{m} = (m_1, m_2, m_3)$. System (1.1) is used to model the evolution of a Newtonian gaseous star ([4]). In the study of time-independent solutions of system (1.1), there are two important cases, non-rotating stars and rotating stars. A non-rotating star solution is a time-independent spherical symmetric solution of the form $(\rho_N, 0, \Phi_N)(x)$ (the velocity is zero), with $\Phi_N(x) = -\rho_N * \frac{1}{|x|}$. A rotating star solution models a star rotating around the x_3-axis ($x = (x_1,\ x_2,\ x_3)$) with prescribed angular momentum (per unit mass), or angular velocity. The existence and properties of stationary non-rotating star solutions is classical (cf. [4]). In contrast, the study for rotating stars is more challenging and of significance in both astrophysics and mathematics. A rigorous mathematical theory for rotating stars of compressible fluids was initiated by Auchmuty & Beals ([1]) in 1971. The existence and properties of rotating star solutions were obtained by Auchmuty & Beals ([1]), Auchmuty([2]), Caffarelli & Friedman ([3]), Friedman & Turkington([13], [14]), Li([21]), Chanillo & Li([5]), and Luo & Smoller ([25]). In [26], McCann proved an existence result for rotating binary stars.

The existence of rotating star solutions of compressible fluids was first obtained by Auchmuty & Beals ([1]) who formulated this problem as a variational problem of finding a minimizer of the energy functional $F(\rho)$, (which will be defined in Section 2), in the class of functions $W_{M,S} = W_M \cap W_S$, where W_M is the set of integrable functions $\rho : \mathbb{R}^3 \to \mathbb{R}^+$ which are a.e. non-negative, axi-symmetric, of total mass $M = \int_{\mathbb{R}^3} \rho(x) dx$, and having a finite rotational kinetic energy (precise statements can be found in Section 2). W_S is defined by

$$W_S = \{\rho : \mathbb{R}^3 \to \mathbb{R}^+,\ \rho(x_1, x_2, -x_3) = \rho(x_1, x_2, x_3),\ x_i \in \mathbb{R},\ i = 1, 2, 3\}. \tag{1.3}$$

In this paper, we first give a proof of the existence of a minimizer of the energy functional $F(\rho)$ in the wider class of functions W_M. Our proof is quite different from that in [1]. As in [1], the main difficulty in the proof is the loss of compactness due to the unboundedness of $\mathbb{R}^3$.

The method in [1] is to minimize the functional F on $W_R = \{\rho \in W_{M,S}, \rho(x) = 0 \ \ |x| > R\}$ and to obtain some uniform estimates on the support of the minimizer. Our method is to use the concentration-compactness method due to P. L. Lions ([24]), which was also used in [29] to prove the existence of non-rotating star solutions. The reason that we seek minimizers in W_M instead of $W_{M,S}$ is that we want to discuss the full stability problem dynamically in a more general context with less restrictions on the symmetry of solutions.

The dynamical stability of these steady-state solutions is an important question. The linearized stability and instability for non-rotating stars and rotating stars were discussed by Lin ([23]), Lebovitz ([19]) and Lebovitz & Lifschitz ([20]). The nonlinear dynamical stability of *non-rotating* star solutions was studied by Rein ([30]) via an energy-Casimir technique. It should mentioned here that the energy-Casimir technique was used in [16] to study the stability problem in stellar dynamics. Roughly speaking, for $p(\rho) = \rho^\gamma$, the result in [30] says that if the initial data of the Euler-Poisson equations (1.1) is close to the non-rotating star solution in some topology, then the solution of (1.1) with the same total mass as the non-rotating star, stays close to the non-rotating solution in the same topology as long as the solution preserves both the energy $E(t)$ which is defined by

$$E(t) = \int_{\mathbb{R}^3} \left(\frac{p(\rho)}{\gamma - 1} + \frac{1}{2}\rho|\mathbf{v}|^2 \right)(x,t)dx - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\Phi|^2(x,t)dx, \tag{1.4}$$

and the total mass $\int_{\mathbb{R}^3} \rho(x,t)dx$. An interesting feature of the energy is that it has both positive and negative parts, making the analysis difficult. For solutions of (1.1) without shock waves, energy is conserved. For solutions with shock waves, the energy $E(t)$ is non-increasing due to the entropy condition associated with shock waves (cf. [18] and [32]). In this paper we extend the above nonlinear stability results to *rotating* stars.

As in the non-rotating star case ([30]), our nonlinear stability result is in the class of solutions having the same total mass as that of the rotating steady-state solution. For solutions with different total masses, we investigate the nonlinear dynamical stability of a solution $\bar{u} = (\bar{\rho}, \bar{\mathbf{v}}, \bar{\Phi}) \in W^{1,\infty}_{loc}$, (which includes both rotating and non-rotating stars), in the context of weak entropy solutions, for more general perturbations not necessarily having the same mass as $\bar{u}$, under some assumptions on the L^∞-norm and the support of the solutions. This is achieved by using the techniques of relative entropies together with a careful analysis of the gravitational energy; i.e., the negative part in the total energy $E(t)$. It should be mentioned here that the method of relative entropies was used by Dafermos ([9]) and Chen/Frid [6]) to study the stability and behavior of solutions of hyperbolic conservation laws. The main difficulty in applying this method to the the Euler-Poisson equations (1.1) is again due to the non-definiteness of the energy density. We also give a uniform a priori estimate for the weak solutions of Cauchy problem of (1.1) satisfying the entropy conditions.

This paper is organized as follows: in Section 2, we prove the existence of rotating star solutions which are the minimizers of an energy functional F in W_M with prescribed total mass and angular momentum with finite rotational kinetic energy. We also derive some properties concerning the minimizing sequence. These properties are interesting, and are important for our stability analysis. In Section 3, we prove our nonlinear stability result for rotating stars. Section 4 is devoted to the stability result for the entropy weak solutions and in Section 5, we obtain uniform in time a priori estimates for entropy weak solutions.

Throughout this paper, for simplicity of presentation, we assume that the pressure function $p(\rho)$ satisfies the usual γ-law,

$$p(\rho) = \rho^\gamma, \ \ \rho \geq 0, \tag{1.5}$$

for some $\gamma > 1$. We now introduce some notation which will be used throughout this paper. We use $\int$ to denote $\int_{\mathbb{R}^3}$, and use $||\cdot||_q$ to denote $||\cdot||_{L^q(\mathbb{R}^3)}$. For any point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, let

$$r(x) = \sqrt{x_1^2 + x_2^2}, \ \ z(x) = x_3, \ \ B_R(x) = \{y \in \mathbb{R}^3, \ \ |y - x| < R\}. \tag{1.6}$$

For any function $f \in L^1(\mathbb{R}^3)$, we define the operator B by

$$Bf(x) = \int \frac{f(y)}{|x-y|} dy = f * \frac{1}{|x|}. \tag{1.7}$$

Also, we use ∇ to denote the spatial gradient, i.e., $\nabla = \nabla_x = (\partial_{x_1}, \ \partial_{x_2}, \ \partial_{x_3})$. C will denote a generic positive constant.

2 Existence of Rotating Star Solutions

A rotating star solution $(\tilde{\rho}, \tilde{\mathbf{v}}, \tilde{\Phi})(r, z)$, where $r = \sqrt{x_1^2 + x_2^2}$ and $z = x_3$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, is an *axi-symmetric* time-independent solution of system (1.1), which models a star rotating about the x_3-axis. Suppose the angular momentum (per unit mass), $J(m_{\tilde{\rho}}(r))$ is prescribed, where

$$m_{\tilde{\rho}}(r) = \int_{\sqrt{x_1^2+x_2^2}<r} \tilde{\rho}(x)dx = \int_0^r 2\pi s \int_{-\infty}^{+\infty} \tilde{\rho}(s, z) ds dz, \tag{2.1}$$

is the mass in the cylinder $\{x = (x_1, x_2, x_3) : \sqrt{x_1^2 + x_2^2} < r\}$, and J is a given function. In this case, the velocity field $\tilde{\mathbf{v}}(x) = (v_1, v_2, v_3)$ takes the form

$$\tilde{\mathbf{v}}(x) = (-\frac{x_2 J(m_{\tilde{\rho}}(r))}{r^2}, \frac{x_1 J(m_{\tilde{\rho}}(r))}{r^2}, 0).$$

Substituting this in (1.1), we find that $\tilde{\rho}(r, z)$ satisfies the following two equations:

$$\begin{cases} \partial_r p(\tilde{\rho}) = \tilde{\rho} \partial_r (B\tilde{\rho}) + \tilde{\rho} L(m_{\tilde{\rho}}(r) r^{-3}, \\ \partial_z p(\tilde{\rho}) = \tilde{\rho} \partial_z (B\tilde{\rho}), \end{cases} \tag{2.2}$$

where the operator B is defined in (1.7), and

$$L(m_{\tilde{\rho}}) = J^2(m_{\tilde{\rho}})$$

is the square of the angular momentum. For any function $\rho \geq 0$ and $\gamma > 1$, we define

$$A(\rho) = \frac{p(\rho)}{\gamma - 1} = \frac{\rho^\gamma}{\gamma - 1}. \tag{2.3}$$

It is easy to verify that (cf. [1]) (2.2) is equivalent to

$$A'(\tilde{\rho}(x)) + \int_{r(x)}^{\infty} L(m_{\tilde{\rho}}(s)s^{-3}ds - B\tilde{\rho}(x) = \lambda, \qquad \text{where } \tilde{\rho}(x) > 0, \tag{2.4}$$

for some constant λ. Here $r(x)$ and $z(x)$ are as in (1.6). In [1], Auchmuty and Beals formulated the problem of finding solutions of (2.4) as the following variational problem. First, let M be a positive constant and let W_M be the set of functions ρ defined by (cf. (1.5)),

$$\begin{aligned} W_M =& \{\rho : \mathbb{R}^3 \to \mathbb{R},\ \rho \text{ is axisymmetric},\ \rho \geq 0, a.e.,\ \rho \in L^1(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3), \\ & \int \rho(x)dx = M,\ \int \frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}dx < +\infty.\} \end{aligned}$$

For $\rho \in W_M$, we define the **energy functional** F by

$$\begin{aligned} F(\rho) &= \int [A(\rho(x)) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2} - \frac{1}{2}\rho(x) \cdot B\rho(x)]dx \\ &= \int [A(\rho(x)) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}]dx - \frac{1}{8\pi}||\nabla B\rho||_2^2. \end{aligned} \tag{2.5}$$

($\frac{1}{8\pi}||B\rho||_2^2 < +\infty$ follows from $\rho \in L^1(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$ and Lemma 2.3 if $\gamma \geq 4/3$.) In (2.5), the first term denotes the potential energy, the middle term denotes the rotational kinetic energy and the third term is the gravitational energy. Assume that the function $L \in C^1[0, M]$ and satisfies

$$L(0) = 0,\ L(m) \geq 0,\ for\ 0 \leq m \leq M. \tag{2.6}$$

Auchmuty and Beals (cf. [1]) proved the existence of a minimizer of the functional $F(\rho)$ in the class of functions $W_{M,S} = W_M \cap W_S$, where

$$W_S = \{\rho : \mathbb{R}^3 \to \mathbb{R},\ \rho(x_1, x_2, -x_3) = \rho(x_1, x_2, x_3),\ x_i \in \mathbb{R}, i = 1,\ 2,\ 3\}. \tag{2.7}$$

Their result is given in the following theorem.

Theorem 2.1. *([1]). If $\gamma > 4/3$ and (2.6) holds, then there exists a function $\hat{\rho}(x) \in W_{M,S}$ which minimizes $F(\rho)$ in $W_{M,S}$. Moreover, if*

$$G = \{x \in \mathbb{R}^3 :\ \hat{\rho}(x) > 0\}, \tag{2.8}$$

Then $\bar{G}$ is a compact set in $\mathbb{R}^3$, and $\hat{\rho} \in C^1(G) \cap C^\beta(\mathbb{R}^3)$ for some $0 < \beta < 1$. Furthermore, there exists a constant $\mu < 0$ such that

$$\begin{cases} A'(\hat{\rho}(x)) + \int_{r(x)}^\infty L(m_{\hat{\rho}}(s)s^{-3}ds - B\hat{\rho}(x) = \mu, & x \in G, \\ \int_{r(x)}^\infty L(m_{\hat{\rho}}(s)s^{-3}ds - B\hat{\rho}(x) \geq \mu, & x \in \mathbb{R}^3 - G. \end{cases} \tag{2.9}$$

In this paper, we are interested in the minimizer of functional F in the *larger* class W_M. By the same argument as in [1], it is easy to prove the following theorem on the regularity of the minimizer.

Theorem 2.2. *Let $\tilde{\rho}$ be a minimizer of the energy functional F in W_M and let*

$$\Gamma = \{x \in \mathbb{R}^3 : \ \tilde{\rho}(x) > 0\}. \tag{2.10}$$

If $\gamma > 6/5$, then $\tilde{\rho} \in C(\mathbb{R}^3) \cap C^1(\Gamma)$. Moreover, there exists a constant λ such that

$$\begin{cases} A'(\tilde{\rho}(x)) + \int_{r(x)}^{\infty} L(m_{\tilde{\rho}}(s)s^{-3}ds - B\tilde{\rho}(x) = \lambda, \qquad x \in \Gamma, \\ \int_{r(x)}^{\infty} L(m_{\tilde{\rho}}(s)s^{-3}ds - B\tilde{\rho}(x) \geq \lambda, \qquad x \in \mathbb{R}^3 - \Gamma. \end{cases} \tag{2.11}$$

We call such a minimizer $\tilde{\rho}$ a *rotating star* solution with total mass M and angular momentum $\sqrt{L(m)}$.

In this paper, we prove the existence of a minimizer for the functional F in the class W_M. For this purpose, in addition to (2.6), we require that L satisfies the following conditions:

$$L(am) \geq a^{4/3}L(m), \ 0 < a \leq 1, \ 0 \leq m \leq M, \tag{2.12}$$

$$L'(m) \geq 0, \qquad 0 \leq m \leq M. \tag{2.13}$$

Remark 1. Condition (2.13) is called the Sölberg stability criterion, see [33, Section 7.3]. This condition was also used by Auchmuty in [2] for the study of global branching of rotating star solutions.

Our main result in this section is the following theorem.

Theorem 2.3. *Suppose that $\gamma > 4/3$ and the square of the angular momentum L satisfies (2.6), (2.12) and (2.13). Then the following hold:*
(1) the functional F is bounded below on W_M and $\inf_{W_M} F(\rho) < 0$,
(2) if $\{\rho^i\} \subset W_M$ is a minimizing sequence for the functional F, then there exist a sequence of vertical shifts $a_i\mathbf{e_3}$ ($a_i \in \mathbb{R}$, $\mathbf{e_3} = (0, 0, 1)$), a subsequence of $\{\rho^i\}$, (still labeled $\{\rho^i\}$), and a function $\tilde{\rho} \in W_M$, such that for any $\epsilon > 0$ there exists $R > 0$ with

$$\int_{a_i\mathbf{e_3}+B_R(0)} \rho^i(x)dx \geq M - \epsilon, \quad i \in \mathbb{N}, \tag{2.14}$$

and

$$T\rho^i(x) := \rho^i(x + a_i\mathbf{e_3}) \rightharpoonup \tilde{\rho}, \ \text{weakly in } L^\gamma(\mathbb{R}^3), \ \text{as } i \to \infty. \tag{2.15}$$

Moreover (3)

$$\nabla B(T\rho^i) \to \nabla B(\tilde{\rho}) \ \text{strongly in } L^2(\mathbb{R}^3), \ \text{as } i \to \infty. \tag{2.16}$$

(4) $\tilde{\rho}$ is a minimizer of F in W_M.

Thus $\tilde{\rho}$ is a rotating star solution with total mass M and angular momentum $\sqrt{L}$.

Remark 2. It is easy to verify that the functional F is invariant under any vertical shift, i.e., if $\rho(\cdot) \in W_M$, then $\bar{\rho}(x) =: \rho(x + a\mathbf{e_3}) \in W_M$ and $F(\bar{\rho}) = F(\rho)$ for any $a \in \mathbb{R}$. Therefore, if $\{\rho^i\}$ is a minimizing sequence of F in W_M, then $\{T\rho^i\}$ defined in (2.15) is also a minimizing sequence in W_M.

Remark 3. In [13], [14] and [5], the diameter estimate of rotating star solutions with the symmetry $\tilde{\rho}(r, -z) = \tilde{\rho}(r, z)$ was obtained. The ideas and techniques developed in [13], [14] and [5] should also be applied to obtain the diameter estimates for the rotating star solutions in Theorem 2.3. Due to the length of this paper, we leave this issue for the future study.

Theorem 2.3 is proved in a sequence of lemmas. We first give some inequalities which will be used later. We begin with Young's inequality (see [17], p. 146.)

Lemma 2.1. *If $f \in L^p \cap L^r$, $1 \le p < q < r \le +\infty$, then*

$$||f||_q \le ||f||_p^a ||f||_r^{1-a}, \qquad a = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}. \tag{2.17}$$

The following two lemmas are proved in [1].

Lemma 2.2. *Suppose the function $f \in L^1(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$. If $1 < q \le 3/2$, then $Bf =: f * \frac{1}{|x|}$ is in $L^r(\mathbb{R}^3)$ for $3 < r < 3q/(3-2q)$, and*

$$||Bf||_r \le C\left(||f||_1^b ||f||_q^{1-b} + ||f||_1^c ||f||_q^{1-c}\right), \tag{2.18}$$

for some constants $C > 0$, $0 < b < 1$, and $0 < c < 1$. If $q > 3/2$, then $Bf(x)$ is a bounded continuous function, and satisfies (2.18) with $r = \infty$.

Lemma 2.3. *For any function $f \in L^1(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$, if $\gamma \ge 4/3$, then $\nabla Bf \in L^2(\mathbb{R}^3)$. Moreover,*

$$|\int f(x)Bf(x)dx| = \frac{1}{4\pi}||\nabla Bf||_2^2 \le C\left(\int |f|^{4/3}(x)dx\right)\left(\int |f|(x)dx\right)^{2/3}, \tag{2.19}$$

for some constant C.

Throughout this paper, we assume the function L, the square of the angular momentum satisfies conditions (2.6), (2.12)and (2.13). Let

$$f_M = \inf_{\rho \in W_M} F(\rho). \tag{2.20}$$

We begin our analysis with the following lemma.

Lemma 2.4. *Suppose $\gamma > 4/3$. If $\rho \in W_M$, then there exist two positive constants C_1 and C_2 depending only on γ and M such that*

$$\int [\rho^\gamma(x) + \frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}]dx \le C_1 F(\rho) + C_2. \tag{2.21}$$

This implies

$$f_M > -\infty,$$

where f_M is defined in (2.20).

Proof. Using (2.19), we have, for $\rho \in W_M$,

$$\begin{aligned} F(\rho) &= \int [A(\rho) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2} - \frac{1}{2}\rho B\rho]dx \\ &\ge \int [A(\rho) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}]dx - C\int \rho^{4/3}dx(\int \rho dx)^{2/3} \\ &= \int [A(\rho) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}]dx - CM^{2/3}\int \rho^{4/3}dx. \end{aligned} \tag{2.22}$$

Taking $p = 1$, $q = 4/3$, $r = \gamma$, and $a = \frac{\frac{3}{4}\gamma - 1}{\gamma - 1}$ in Young's inequality (2.17), we obtain,

$$||\rho||_{4/3} \le ||\rho||_1^a ||\rho||_\gamma^{1-a} = M^a ||\rho||_\gamma^{1-a}. \tag{2.23}$$

This is

$$\int \rho^{4/3}dx \le M^{\frac{4}{3}a}(\int \rho^\gamma dx)^b, \tag{2.24}$$

where $b = \frac{1}{3(\gamma-1)}$. Since $\gamma > 4/3$, we have $0 < b < 1$. Therefore, (2.22) and (2.24) imply

$$\int [A(\rho) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}]dx \le F(\rho) + C(\gamma-1)^b M^{\frac{4}{3}a+\frac{2}{3}}(\int A(\rho)dx)^b. \tag{2.25}$$

Using (2.24) and the inequality (cf. [17] p. 145)

$$\alpha\beta \le \epsilon\alpha^s + \epsilon^{-t/s}\beta^t, \tag{2.26}$$

if $s^{-1} + t^{-1} = 1$ $(s, t > 1)$ and $\epsilon > 0$, since $b < 1$, we can bound the last term in (2.25) by $\frac{1}{2}\int A(\rho)dx + C_2$, where C_2 is a constant depending only on M and γ (we can take $\epsilon = 1/2$ and $s = 1/b$ and $t = (1 - s^{-1})^{-1}$ in (2.26) since $s > 1$ due to $0 < b < 1$). This implies (2.21). □

We also need the following lemma.

Lemma 2.5. *Suppose $\gamma > 4/3$. Then*
(a) $f_M < 0$ for every $M > 0$,
(b) if (2.12) holds, then $f_{\bar{M}} \ge (\bar{M}/M)^{5/3} f_M$ for every $M > \bar{M} > 0$.

Proof. It follows from [1] that there exists $\hat{\rho} \in W_{M,S} \subset W_M$ such that $F(\hat{\rho}) = \inf_{\rho \in W_{M,S}} F(\rho)$. By Theorem 2.1, it is easy to verify that the triple $(\hat{\rho}, \hat{\mathbf{v}}, \hat{\Phi})$ is a time-independent solution of the Euler-Poisson equations (1.1) in the region $G = \{x \in \mathbb{R}^3 : \ \hat{\rho}(x) > 0\}$, where $\hat{\mathbf{v}} = (-\frac{x_2 J(m_{\hat{\rho}}(r))}{r}, \frac{x_1 J(m_{\hat{\rho}}(r))}{r}, 0)$ and $\hat{\Phi} = -B\hat{\rho}$. Therefore

$$\nabla_x p(\hat{\rho}) = \hat{\rho}\nabla_x(B\hat{\rho}) + \hat{\rho} L(m_{\hat{\rho}}) r(x)^{-3}\mathbf{e}_r, \ x \in G, \tag{2.27}$$

where $\mathbf{e}_r = (\frac{x_1}{r(x)}, \frac{x_2}{r(x)}, 0)$. Moreover, it is proved in [3] that the boundary ∂G of G is smooth enough to apply the Gauss-Green formula (cf. [12]) on G. Applying the Gauss-Green formula on G and noting that $\hat{\rho}|_{\partial G} = 0$, we obtain,

$$\int_G x \cdot \nabla_x p(\hat{\rho}) dx = -3 \int_G p(\hat{\rho}) dx = -3 \int p(\hat{\rho}) dx. \tag{2.28}$$

By an argument in [33] (used also in [10]), we obtain

$$\int_G x \cdot \hat{\rho}\nabla_x B\hat{\rho} dx = -\frac{1}{2}\int_G \hat{\rho} B\hat{\rho} dx = -\frac{1}{2}\int \hat{\rho} B\hat{\rho} dx. \tag{2.29}$$

(In fact, this can be verified as follows. Let

$$I = \int_G x \cdot \hat{\rho}\nabla_x B\hat{\rho} dx = -\int_G \hat{\rho}(x) \int_G \frac{\rho(y)(x-y)\cdot x}{|x-y|^3} dy dx.$$

Then

$$\begin{aligned} I &= -\int_G \hat{\rho}(x) \int_G \frac{\hat{\rho}(y)(x-y)\cdot(x-y)}{|x-y|^3} dy dx - \int_G \hat{\rho}(x) \int_G \frac{\rho(y)(x-y)\cdot y}{|x-y|^3} dy dx \\ &= -\int_G \hat{\rho}(x) \int_G \frac{\hat{\rho}(y)(x-y)\cdot(x-y)}{|x-y|^3} dy dx - I \\ &= -\int_G \hat{\rho} B\hat{\rho} dx - I, \end{aligned} \tag{2.30}$$

which is (2.29).) Next, since $x \cdot \mathbf{e}_r = r(x)$, we have

$$\begin{aligned} &\int_G x \cdot \hat{\rho}(x) L(m_{\hat{\rho}}(r(x)) r^{-3}(x)\mathbf{e}_r dx \\ &= \int_G \hat{\rho}(x) L(m_{\hat{\rho}}(r(x)) r^{-2}(x) dx \\ &= \int \hat{\rho}(x) L(m_{\hat{\rho}}(r(x)) r^{-2}(x) dx. \end{aligned} \tag{2.31}$$

Therefore, from (2.28)-(2.31) we have

$$-3\int p(\hat{\rho}) dx = -\frac{1}{2}\int \hat{\rho} B\hat{\rho} dx + \int \hat{\rho}(x) L(m_{\hat{\rho}}(r(x)) r^{-2}(x) dx, \tag{2.32}$$

so that

$$F(\hat{\rho}) = \frac{4-3\gamma}{\gamma-1}\int p(\hat{\rho}) dx - \frac{1}{2}\int \hat{\rho}(x) L(m_{\hat{\rho}}(r(x)) r^{-2}(x) dx.$$

Thus, if $\gamma > 4/3$, $F(\hat{\rho}) < 0$ since $L(m) \geq 0$ for $0 \leq m \leq M$. Since $\hat{\rho} \in W_{M,S} \subset W_M$, then $inf_{\rho \in W_M} F(\rho) < 0$. This completes the proof of part (a).
The proof of part (b) follows from a scaling argument as in [29]. Taking $b = (M/\bar{M})^{1/3}$ and letting $\bar{\rho}(x) = \rho(bx)$ for any $\rho \in W_M$. It is easy to verify that $\bar{\rho} \in W_{\bar{M}}$ and that the following identities hold,

$$\int \bar{\rho} B \bar{\rho} dx = b^{-5} \int \rho B \rho dx, \tag{2.33}$$

$$\int A(\bar{\rho}) dx = b^{-3} \int A(\rho) dx. \tag{2.34}$$

Moreover, for $r \geq 0$,

$$\begin{aligned} m_{\bar{\rho}}(r) &= 2\pi \int_0^r s \int_{-\infty}^{\infty} \bar{\rho}(s,z) ds dz \\ &= 2\pi \int_0^r s \int_{-\infty}^{\infty} \rho(bs, bz) ds dz \\ &= \frac{1}{b^3} 2\pi \int_0^{br} s' \int_{-\infty}^{\infty} \rho(s', z') ds' dz' \\ &= \frac{1}{b^3} m_\rho(br). \end{aligned} \tag{2.35}$$

Since L satisfies (2.12) and $b > 1$, we have

$$L(m_{\bar{\rho}}(r)) \geq \frac{1}{b^4} L(m_\rho(br)). \tag{2.36}$$

Thus,

$$\begin{aligned} \int \frac{\bar{\rho}(x) L(m_{\bar{\rho}}(r(x)))}{r(x)^2} dx &\geq \frac{1}{b^4} \int_0^{+\infty} \frac{2\pi r}{r^2} L(m_\rho(br)) \int_{-\infty}^{\infty} \rho(br, bz) dz dr \\ &= \frac{1}{b^5} \int_0^{+\infty} \frac{2\pi r'}{r'^2} L(m_\rho(r')) \int_{-\infty}^{\infty} \rho(r', z') dz' dr' \\ &= \frac{1}{b^5} \int \frac{\rho(x) L(m_{\bar{\rho}}(r(x)))}{r(x)^2} dx. \end{aligned} \tag{2.37}$$

Therefore, since $b \geq 1$, it follows from (2.33)-(2.37) that

$$\begin{aligned} F(\bar{\rho}) &\geq b^{-3} \int A(\rho) dx - \frac{b^{-5}}{2} \int \rho B \rho dx + \frac{b^{-5}}{2} \int \frac{\rho(x) L(m_{\bar{\rho}}(r(x)))}{r(x)^2} dx \\ &\geq b^{-5} \left(\int A(\rho) dx - \frac{1}{2} \int \rho B \rho dx + \frac{1}{2} \int \frac{\rho(x) L(m_{\bar{\rho}}(r(x)))}{r(x)^2} dx \right) \\ &= (\bar{M}/M)^{5/3} F(\rho). \end{aligned} \tag{2.38}$$

Since $\rho \to \bar{\rho}$ is one-to-one between W_M and $W_{\bar{M}}$, this proves part (b).

□

The following lemma gives the boundedness of a minimizing sequence of F in $L^\gamma(\mathbb{R}^3)$.

Lemma 2.6. *Suppose $\gamma > 4/3$. Let $\{\rho^i\} \subset W_M$ be a minimizing sequence of F. Then $\{\rho^i\}$ is bounded in $L^\gamma(\mathbb{R}^3)$, and moreover, the rotating kinetic energy*

$$\frac{1}{2}\int \frac{\rho^i(x)L(m_{\rho^i}(r(x)))}{r(x)^2}$$

is also uniformly bounded.

Proof. By Lemma 2.4, we have

$$\int [(\rho^i)^\gamma(x) + \frac{\rho^i(x)L(m_{\rho^i}(r(x)))}{r(x)^2}]dx \leq C_1 F(\rho^i) + C_2,\ i \geq 1. \tag{2.39}$$

The lemma follows from this and Part a) in Lemma 2.5. □

Lemma 2.7. *Suppose $\gamma > 4/3$. Let $\{\rho^i\} \subset W_M$ be a minimizing sequence for F. Then there exist constants $r_0 > 0$, $\delta_0 > 0$, $i_0 \in \mathbf{N}$ and $x^i \in \mathbb{R}^3$ with $r(x^i) \leq r_0$, such that*

$$\int_{B_1(x^i)} \rho^i(x)dx \geq \delta_0,\ i \geq i_0. \tag{2.40}$$

Proof. The proof of this lemma essentially follows from those arguments in [1] with slight modification. First, since $\lim_{i\to\infty} F(\rho^i) \to f_M$ and $f_M < 0$ (see part (a) of Lemma 2.5), for large i,

$$-\frac{f_M}{2} \leq -F(\rho^i) \leq \frac{1}{2}\int \rho^i B\rho^i dx \leq \frac{1}{2}M||B\rho^i||_\infty. \tag{2.41}$$

So

$$||B\rho^i||_\infty \geq -f_M/M,\ \text{for large } i. \tag{2.42}$$

For any i, let

$$\delta_i = \sup_{x\in\mathbb{R}^3} \int_{|y-x|<1} \rho^i(y)dy. \tag{2.43}$$

Now

$$\begin{aligned} B\rho^i(x) &= \int |y-x|^{-1}\rho^i(y)dy \\ &= \int_{|y-x|<1} + \int_{1<|y-x|<r} + \int_{|y-x|>r} \\ &=: B_1 + B_2 + B_3, \end{aligned} \tag{2.44}$$

and $B_3 \leq Mr^{-1}$. The shell $1 < |y-x| < r$ can be covered by at most Cr^3 balls of radius 1, so $B_2 \leq C\delta_i r^3$. In order to estimate B_1, we apply (2.18) to $\chi_{\{y||y-x|<1\}}\rho^i(y)$, where χ is the indicator function. This gives $B_1 \leq C(\delta_i^a + \delta_i^b)$, where $a, b > 0$. It follows that we could choose r so large that the above estimates give $B\rho^i < -f_M/M$ *if δ_i were small enough.* This would contradict (2.42). So there exists $\delta_0 > 0$ such that $\delta_i \geq \delta_0$ for large i. Thus, as i is large, there exists $x^i \in \mathbb{R}^3$ and $i_0 \in \mathbb{N}$ such that

$$\int_{B_1(x^i)} \rho^i(x)dx \geq \delta_0, \ i \geq i_0. \tag{2.45}$$

We now prove that there exists $r_0 > 0$ independent of i such that those x^i must satisfy $r(x^i) \leq r_0$ for i large. Namely, since ρ^i has mass at least δ_0 in the unit ball centered at x^i, and is axially symmetric, it has mass $\geq Cr(x^i)\delta_0$ in the torus obtained by revolving this ball around x_3-axis (or z-axis).Therefore $r(x^i) \leq (C\delta_0)^{-1}M$. □

In order to prove Theorem 2.3, we will need the following lemma which is proved in [29], and uses a concentration-compactness argument.

Lemma 2.8. *Suppose $\gamma > 4/3$. Let $\{f^i\}$ be a bounded sequence in $L^\gamma(\mathbb{R}^3)$ and suppose*

$$f^i \rightharpoonup f^0 \ \ \textit{weakly in } L^\gamma(\mathbb{R}^3).$$

Then
(a) For any $R > 0$,

$$\nabla B(\chi_{B_R(0)} f^i) \to \nabla B(\chi_{B_R(0)} f^0) \ \ \textit{strongly in } L^2(\mathbb{R}^3),$$

where χ is the indicator function.
(b) If in addition $\{f^i\}$ is bounded in $L^1(\mathbb{R}^3)$, $f^0 \in L^1(\mathbb{R}^3)$, and for any $\epsilon > 0$ there exist $R > 0$ and $i_0 \in \mathbf{N}$ such that

$$\int_{|x|>R} |f^i(x)|dx < \epsilon, \qquad i \geq i_0, \tag{2.46}$$

then

$$\nabla B f^i \to \nabla B f^0 \ \textit{strongly in } L^2(\mathbb{R}^3).$$

Before giving the proof of Theorem 2.3, we first outline the main steps. In step 1, we first show (2.15) and (2.16). In step 2 we show that if $\tilde{\rho}$ is a weak limit in $L^\gamma(\mathbb{R}^3)$ of $\{T\rho^i\}$, then $m_{\tilde{\rho}}(r)$ is a continuous function of r for all $r \geq 0$. The third step is to prove that F is lower semi-continuous with respect to the weak topology in $L^\gamma(\mathbb{R}^3)$.

Proof of Theorem 2.3

Step 1. We prove (2.16), and apply Lemma 2.8 to prove (2.14). We begin with a splitting as in [29]. For $\rho \in W_M$, for any $0 < R_1 < R_2$, we have

$$\rho = \rho\chi_{|x|\leq R_1} + \rho\chi_{R_1<|x|\leq R_2} + \rho\chi_{|x|>R_2} =: \rho_1 + \rho_2 + \rho_3, \tag{2.47}$$

where χ is the indicator function. It is easy to verify that

$$\int A(\rho)dx = \sum_{j=1}^{3} \int A(\rho_j)dx, \tag{2.48}$$

and

$$\int \rho B\rho dx = \sum_{j=1}^{3} \int \rho_j B\rho_j dx + I_{12} + I_{13} + I_{23}, \tag{2.49}$$

where

$$I_{ij} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x-y|^{-1} \rho_i(x)\rho_j(y)dxdy, \qquad 1 \le i < j \le 3.$$

Also,

$$\begin{aligned} \int \frac{\rho(x)L(m_\rho(r(x))}{r^2(x)}dx &= \sum_{j=1}^{3} \int \frac{\rho_j(x)L(m_{\rho_j}(r(x))}{r^2(x)}dx \\ &+ \sum_{j=1}^{3} \int \frac{\rho_j(x)(L(m_\rho(r(x)) - L(m_{\rho_j}(r(x))}{r^2(x)}dx. \end{aligned} \tag{2.50}$$

It follows from (2.47)-(2.50) that

$$\begin{aligned} F(\rho) = &\sum_{j=1}^{3} F(\rho_j) - \frac{1}{2} \sum_{1\le i<j\le 3} I_{ij} \\ &+ \frac{1}{2}\sum_{j=1}^{3} \int \frac{\rho_j(x)(L(m_\rho(r(x)) - L(m_{\rho_j}(r(x))}{r^2(x)}dx. \end{aligned} \tag{2.51}$$

Since $\rho \ge \rho_j$, we have $m_\rho(r) \ge m_{\rho_j}(r)$ for any $r \ge 0$ and $j = 1, 2, 3$. By (2.13),

$$F(\rho) \ge \sum_{j=1}^{3} F(\rho_j) - \frac{1}{2} \sum_{1\le i<j\le 3} I_{ij}. \tag{2.52}$$

Using (2.52) and Lemma 2.5, by the same argument as in the proof of Theorem 3.1 in [29], we can show that

$$f_M - F(\rho) \le Cf_M M_1 M_3 + C(R_2^{-1} + ||\rho||_\gamma^{(q+1)/6} ||\nabla B\rho_2||_2), \tag{2.53}$$

by choosing $R_2 > 2R_1$ in the splitting (2.47), where $M_1 = \int \rho_1(x)dx = \int_{|x|\le R_1} \rho(x)dx$, $M_3 = \int \rho_3(x)dx = \int_{|x|>R_2} \rho(x)dx$ and $q = 1/(\gamma - 1)$. Let $\{\rho^i\}$ be a minimizing sequence of F in W_M. By Lemma 2.7, we know that there exists $i_0 \in \mathbf{N}$ and $\delta_0 > 0$ independent of i such that

$$\int_{B_1(x^i)} \rho^i(x)dx \ge \delta_0, \qquad i \ge i_0 \tag{2.54}$$

for some $x^i \in \mathbb{R}^3$ with $r(x^i) \le r_0$ for some constant $r_0 > 0$ independent of i. Let $a_i = z(x^i)$ and $R_0 = r_0 + 1$, then (2.54) implies

$$\int_{a_i\mathbf{e_3}+B_{R_0(0)}} \rho^i(x)dx \ge \delta_0, \qquad if\ i \ge i_0, \tag{2.55}$$

where $\mathbf{e_3} = (0,0,1)$. Having proved (2.55), we can follow the argument in the proof of Theorem 3.1 in [29] to verify (2.46) for

$$f^i(x) = T\rho^i(x) =: \rho^i(\cdot + a_i\mathbf{e_3})$$

by using (2.52) and (2.55) and choosing suitable R_1 and R_2 in the splitting (2.47). We sketch this as follows. The sequence $T\rho^i =: \rho^i(\cdot + a_i\mathbf{e_3})$, $i \ge i_0$, is a minimizing sequence of F in W_M (see Remark 2 after Theorem 2.3). We rewrite (2.55) as

$$\int_{B_R(0)} T\rho^i(x)dx \ge \delta_0,\ i \ge i_0. \tag{2.56}$$

Applying (2.53) with $T\rho^i$ replacing ρ, and noticing that $\{T\rho^i\}$ is bounded in $L^\gamma(\mathbb{R}^3)$ (see Lemma 2.6), we obtain, if $R_2 > 2R_1$,

$$-Cf_M M_1^i M_3^i \le C(R_2^{-1} + ||\nabla BT\rho_2^i||_2) + F(T\rho^i) - f_M, \tag{2.57}$$

where $M_1^i = \int T\rho_1^i(x)dx = \int_{|x|<R_1} T\rho^i(x)dx,$, $M_3^i = \int T\rho_3^i(x)dx = \int_{|x|>R_2} T\rho^i(x)dx$ and $T\rho_2^i = \chi_{R_1<|x|\le R_2} T\rho^i$. Since $\{T\rho^i\}$ is bounded in $L^\gamma(\mathbb{R}^3)$, there exists a subsequence, still labeled by $\{T\rho^i\}$, and a function $\tilde\rho \in W_M$ such that

$$T\rho^i \rightharpoonup \tilde\rho \text{ weakly in } L^\gamma(\mathbb{R}^3).$$

This proves (2.15). By (2.56), we know that M_1^i in (2.57) satisfies $M_1^i \ge \delta_0$ for $i \ge i_0$ by choosing $R_1 \ge R_0$ where R_0 is the constant in (2.56). Therefore, by (2.57) and the fact that $f_M < 0$ (cf. Part (a) in Lemma 2.5) , we have

$$-Cf_M\delta_0 M_3^i \le CR_2^{-1} + C||\nabla B\tilde\rho_2||_2 + C||\nabla BT\rho_2^i - \nabla B\tilde\rho_2||_2) + F(T\rho^i) - f_M, \tag{2.58}$$

where $\tilde\rho_2 = \chi_{|x|>R_2}\tilde\rho$. Given any $\epsilon > 0$, by the same argument as [29], we can increase $R_1 > R_0$ such that the second term on the right hand side of (2.58) is small, say less than $\epsilon/4$. Next choose $R_2 > 2R_1$ such that the first term is small. Now that R_1 and R_2 are fixed, the third term on the right hand side of (2.58) converges to zero by Lemma 2.8(a). Since $\{T\rho^i\}$ is a minimizing sequence of F in W_M, we can make $F(T\rho^i) - f_M$ small by taking i large. Therefore, for i sufficiently large, we can make

$$M_3^i =: \int_{|x|>R_2} T\rho^i(x)dx < \epsilon. \tag{2.59}$$

This verifies (2.46) in Lemma 2.8 for $f^i = T\rho^i$. Therefore, by Lemma 2.8(b),we have

$$||\nabla BT\rho^i - \nabla B\tilde{\rho}||_2 \to 0, \qquad i \to +\infty. \tag{2.60}$$

This proves (2.16). (2.14) in Theorem 2.3 follows from (2.59) by taking $R = R_2$.

Step 2. Let $\tilde{\rho}$ be a weak limit of a subsequence of $\{T\rho^i\}$ in $L^\gamma(\mathbb{R}^3)$ (we still label the subsequence by $\{T\rho^i\}$). We claim that the mass function

$$m_{\tilde{\rho}}(r) =: \int_{\sqrt{x_1^2+x_2^2}\le r} \tilde{\rho}(x)dx \quad \text{is continuous for } r \ge 0. \tag{2.61}$$

This is proved as follows. By the weak convergence of $\{T\rho^i\}$ and noting that $\int T\rho^i dx = M$, we know that $\int \tilde{\rho} dx = M$. Also, by the lower semicontinuity of norms (cf. [22] p.51) and Lemma 2.6, we have

$$||\tilde{\rho}||_\gamma \le \lim_{i\to\infty}\inf ||T\rho^i||_\gamma = \lim_{i\to\infty}\inf ||\rho^i||_\gamma \le C, \tag{2.62}$$

for some positive constant C. For any $\epsilon > 0$, by the weak convergence and (2.14) which we have already proved, there exists $R > 0$ such that

$$\int_{|x|>R} T\rho^i(x)dx < \epsilon, \qquad i \in \mathbb{N}, \tag{2.63}$$

and

$$\int_{|x|>R} \tilde{\rho}(x)dx = \lim_{i\to\infty} \int_{|x|>R} T\rho^i(x)dx \le \epsilon. \tag{2.64}$$

For any $r \ge 0$ and $r_1 \ge r$,

$$\begin{aligned} &0 \le m_{\tilde{\rho}}(r_1) - m_{\tilde{\rho}}(r) \\ &= \int_{r\le\sqrt{x_1^2+x_2^2}\le r_1} \tilde{\rho}(x)dx \\ &\int_{r\le\sqrt{x_1^2+x_2^2}\le r_1, |x_3|>R} \tilde{\rho}(x)dx + \int_{r\le\sqrt{x_1^2+x_2^2}\le r_1, |x_3|\le R} \tilde{\rho}(x)dx. \end{aligned} \tag{2.65}$$

Since $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : r \le \sqrt{x_1^2 + x_2^2} \le r_1, |x_3| > R\} \subset \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| > R\}$, by (2.64), we have

$$\int_{r\le\sqrt{x_1^2+x_2^2}\le r_1, |x_3|>R} \tilde{\rho}(x)dx < \epsilon. \tag{2.66}$$

By (2.62) and Hölder's inequality,

$$\begin{aligned} &\int_{r\le\sqrt{x_1^2+x_2^2}\le r_1, |x_3|\le R} \tilde{\rho}(x)dx \\ &\le ||\tilde{\rho}||_\gamma \left(meas\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : r \le \sqrt{x_1^2 + x_2^2} \le r_1, |x_3| \le R\} \right)^{1/\gamma'} \\ &\le C[2\pi R(r_1 + r)(r_1 - r)]^{1/\gamma'}, \end{aligned} \tag{2.67}$$

where $meas$ denotes the Lebsgue measure and $\gamma' = (\gamma-1)/\gamma$. Now, if we take $\delta = \min\{\frac{(\epsilon/C)^{\gamma'}}{2\pi R(2r+1)}, 1\}$, then by (2.67), we obtain

$$\int_{r\leq\sqrt{x_1^2+x_2^2}\leq r_1, |x_3|\leq R} \tilde{\rho}(x)dx < \epsilon, \tag{2.68}$$

whenever $0 \leq r_1 - r < \delta$. It follows from (2.65), (2.66) and (2.67), we have

$$|m_{\tilde{\rho}}(r_1) - m_{\tilde{\rho}}(r)| < 2\epsilon, \tag{2.69}$$

whenever $0 \leq r_1 - r < \delta$. This proves that $m_{\tilde{\rho}}(r)$ is continuous from the right for any $r \geq 0$. By the same method, we can show that $m_{\tilde{\rho}}(r)$ is continuous from the left for any $r > 0$. Since $m_{\tilde{\rho}}(0) = 0$, this proves (2.61).

Step 3. Let $\{\rho^i\}$ be a minimizing sequence of the energy functional F, and let $\tilde{\rho}$ be a weak limit of $\{T\rho^i\}$ in $L^\gamma(\mathbb{R}^3)$. We will prove that $\tilde{\rho}$ is a minimizer of F in W_M; that is

$$F(\tilde{\rho}) \leq \lim\inf_{i\to\infty} F(T\rho^i). \tag{2.70}$$

First, by (2.62), we have

$$\int A(\tilde{\rho})dx \leq \lim\inf_{i\to\infty} \int A(T\rho^i)dx. \tag{2.71}$$

We fix a positive number δ and show that

$$\lim_{i\to\infty} \int_{r(x)\geq\delta} \frac{T\rho^i(x)L(m_{T\rho^i}(r(x)) - \tilde{\rho}(x)L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx = 0. \tag{2.72}$$

To see this, we write

$$\begin{aligned} &\int_{r(x)\geq\delta} \frac{(T\rho^i(x)L(m_{T\rho^i}(r(x)) - \tilde{\rho}(x)L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx \\ &= \int_{r(x)\geq\delta} \frac{(T\rho^i(x) - \tilde{\rho}(x))L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx \\ &+ \int_{r(x)\geq\delta} \frac{T\rho^i(x)(L(m_{T\rho^i}(r(x)) - L(m_{\tilde{\rho}}(r(x)))}{r^2(x)}dx. \end{aligned} \tag{2.73}$$

For any $R > 0$, we have

$$\begin{aligned} &\int_{r(x)\geq\delta} \frac{(T\rho^i(x) - \tilde{\rho}(x))L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx \\ &= \int_{r(x)\geq\delta, |x|\leq R} \frac{(T\rho^i(x) - \tilde{\rho}(x))L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx \\ &+ \int_{r(x)\geq\delta, |x|\geq R} \frac{(T\rho^i(x) - \tilde{\rho}(x))L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx. \end{aligned} \tag{2.74}$$

In view of (2.63) and (2.64), for any $\epsilon > 0$, we can choose R such that

$$|\int_{r(x)\geq\delta,|x|\geq R}\frac{(T\rho^i(x)-\tilde{\rho}(x))L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx|\leq\frac{2L(M)\epsilon}{\delta^2}. \tag{2.75}$$

By the weak convergence of $\{T\rho^i\}$ in $L^\gamma(\mathbb{R}^3)$ and the fact that L is defined on a bounded range, $L(m_{\tilde{\rho}}(r(x))\chi_{\{r(x)\geq\delta,|x|\leq R\}}(x)r^{-2}(x)\in L^{\gamma'}(\mathbb{R}^3)$, where as before χ is the indicator function, and $\gamma'=\frac{\gamma}{\gamma-1}$ (satisfying $1/\gamma+1/\gamma'=1$). We have

$$\begin{aligned}&\lim_{i\to\infty}\int_{r(x)\geq\delta,|x|\leq R}\frac{(T\rho^i(x)-\tilde{\rho}(x))L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx\\&=\lim_{i\to\infty}\int(T\rho^i(x)-\tilde{\rho}(x))L(m_{\tilde{\rho}}(r(x))\chi_{\{r(x)\geq\delta,|x|\leq R\}}(x)r^{-2}(x)dx\\&=0,\end{aligned} \tag{2.76}$$

because $T\rho^i$ converges weakly to $\tilde{\rho}$. Since ϵ is arbitrary, (2.75) and (2.76) imply

$$\lim_{i\to\infty}\int_{r(x)\geq\delta}\frac{(T\rho^i(x)-\tilde{\rho}(x))L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx=0. \tag{2.77}$$

We handle the second term in (2.73) as follows. By weak convergence, we know that $m_{T\rho^i}(r)$ converges to $m_{\tilde{\rho}}(r)$ pointwise for $r\geq 0$. Since $m_{T\rho^i}(r)$ and $m_{\tilde{\rho}}(r)$ are non-decreasing functions of r for $r\geq 0$ and $m_{\tilde{\rho}}(r)$ is continuous on $[0,+\infty)$ (see (2.61)), by a variation on Dini's theorem ([31], p.167)*, we know that $m_{T\rho^i}(r)$ converges to $m_{\tilde{\rho}}(r)$ uniformly on the interval $[0,R]$ for any $R>0$. Since $L\in C^1[0,M]$, it follows that $L(m_{T\rho^i}(r))$ converges to $L(m_{\tilde{\rho}}(r))$ uniformly on any interval $[0,R]$. For any $\epsilon>0$, we can fix $R>0$ such that (2.63) and (2.64) hold. Since $L(m_{T\rho^i}(r))$ converges uniformy to $L(m_{\tilde{\rho}}(r))$ on any interval $[0,R]$, we have

$$\lim_{i\to\infty}||L(m_{T\rho^i}(\cdot))-L(m_{\tilde{\rho}}(\cdot))||_{L^\infty[0,R]}=0. \tag{2.78}$$

Let

$$A_\delta=\{x\in\mathbb{R}^3,r(x)\geq\delta\}, \tag{2.79}$$

then we have, using (2.63) and (2.64) that

$$\begin{aligned}&|\int_{r(x)\geq\delta}\frac{T\rho^i(x)(L(m_{T\rho^i}(r(x))-L(m_{\tilde{\rho}}(r(x)))}{r^2(x)}dx|\\&\leq|\int_{A_\delta\cap B_R(0)}\frac{T\rho^i(x)(L(m_{T\rho^i}(r(x))-L(m_{\tilde{\rho}}(r(x)))}{r^2(x)}dx|\\&+|\int_{A_\delta-B_R(0)}\frac{T\rho^i(x)(L(m_{T\rho^i}(r(x))-L(m_{\tilde{\rho}}(r(x)))}{r^2(x)}dx|\\&\leq||L(m_{T\rho^i}(\cdot))-L(m_{\tilde{\rho}}(\cdot))||_{L^\infty[0,R]}\delta^{-2}M+2\delta^{-2}L(M)\epsilon.\end{aligned} \tag{2.80}$$

* We thank Dmitry Khanvinson for pointing out this to us.

Since ϵ is arbitrary, it follows from (2.78) and (2.80) that

$$\lim_{i\to\infty}\int_{r(x)\geq\delta}\frac{T\rho^i(x)(L(m_{T\rho^i}(r(x))-L(m_{\tilde\rho}(r(x)))}{r^2(x)}dx=0. \tag{2.81}$$

This, together with (2.73)and (2.77), implies (2.72). Next, we show that

$$\liminf_{i\to\infty}\int\frac{T\rho^i(x)L(m_{T\rho^i}(r(x))-\tilde\rho(x)L(m_{\tilde\rho}(r(x))}{r^2(x)}dx\geq 0, \tag{2.82}$$

by using (2.72) and the monotone convergence theorem for integrals. In fact we have

$$\begin{aligned}&\int\frac{T\rho^i(x)L(m_{T\rho^i}(r(x))-\tilde\rho(x)L(m_{\tilde\rho}(r(x)))}{r^2(x)}dx\\&=\int\frac{T\rho^i(x)L(m_{T\rho^i}(r(x)))(1-\chi_{A_\delta})}{r^2(x)}dx\\&+\int\frac{[T\rho^i(x)L(m_{T\rho^i}(r(x)))-\tilde\rho(x)L(m_{\tilde\rho}(r(x)))]\chi_{A_\delta}}{r^2(x)}dx\\&+\int\frac{\tilde\rho(x)L(m_{\tilde\rho}(r(x)))(\chi_{A_\delta}-1)}{r^2(x)}dx,\end{aligned} \tag{2.83}$$

where χ is the indicator function, and A_δ is the set defined in (2.79). For any $i\geq 1$,

$$\int\frac{T\rho^i(x)L(m_{T\rho^i}(r(x)))(1-\chi_{A_\delta})}{r^2(x)}dx\geq 0. \tag{2.84}$$

We fix δ, and by (2.72), we know that the second term on the right hand side of (2.83) approaches zero as $i\to\infty$. Therefore, in view of (2.84),

$$\begin{aligned}&\liminf_{i\to\infty}\int\frac{T\rho^i(x)L(m_{T\rho^i}(r(x)))-\tilde\rho(x)L(m_{\tilde\rho}(r(x)))}{r^2(x)}dx\\&\geq\int\frac{\tilde\rho(x)L(m_{\tilde\rho}(r(x)))(\chi_{A_\delta}-1)}{r^2(x)}dx.\end{aligned} \tag{2.85}$$

By the monotone convergence theorem of integrals, we have

$$\lim_{\delta\to 0}|\int\frac{\tilde\rho(x)L(m_{\tilde\rho}(r(x)))(\chi_{A_\delta}-1)}{r^2(x)}dx|=0. \tag{2.86}$$

Letting $\delta\to 0$ in (2.85), gives (2.82). By (2.60), (2.71) and (2.82), we obtain

$$F(\tilde\rho)\leq\liminf_{i\to\infty}F(T\rho^i). \tag{2.87}$$

Since $T\rho^i$ is a minimizing sequence, $\tilde\rho$ is a minimizer of F in W_M. This completes the proof of Theorem 2.3.

3 Nonlinear Stability of Rotating Star Solutions

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We consider the Cauchy problem for (1.1) with the initial data

$$\rho(x,0)=\rho_0(x),\ \ \mathbf{v}(x,0)=\mathbf{v}_0(x). \tag{3.1}$$

We begin by giving the definition of a weak solution.

Definition: Let $\rho\mathbf{v}=\mathbf{m}$. The triple $(\rho,\mathbf{m},\Phi)(x,t)$ $(x\in\mathbb{R}^3,t\in[0,T])$ $(T>0)$ and Φ given by (1.2, with $\rho\geq 0$, $\mathbf{m}$, $\mathbf{m}\otimes\mathbf{m}/\rho$ and $\rho\nabla\Phi$ being in $L^1_{loc}(\mathbb{R}^3\times[0,T])$, is called a *weak solution* of the Cauchy problem (1.1) and (3.1) on $\mathbb{R}^3\times[0,T]$ if for any Lipschitz continuous test functions ψ and $\mathbf{\Psi}=(\psi_1,\psi_2,\psi_3)$ with compact supports in $\mathbb{R}^3\times[0,T]$,

$$\int_0^T\int(\rho\psi_t+\mathbf{m}\cdot\nabla\psi)\,dxdt+\int\rho_0(x)\psi(x,0)dx=0, \tag{3.2}$$

and

$$\int_0^T\int\left(\mathbf{m}\cdot\mathbf{\Psi}_t+\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\cdot\nabla\mathbf{\Psi}\right)dxdt+\int\mathbf{m}_0(x)\mathbf{\Psi}(x,0)dx=\int_0^T\int\rho\nabla\Phi\mathbf{\Psi}dxdt, \tag{3.3}$$

both hold.

For any weak solution, it is easy to verify that the total mass is conserved by using a generalized divergence theorem for L^r functions $(r\geq 1)$ (cf. [7]),

$$\int\rho(x,t)dx=\int\rho(x,0)dx,\qquad t\geq 0. \tag{3.4}$$

The *total energy* of system (1.1) at time t is

$$E(t)=E(\rho(t),\mathbf{v}(t))=\int\left(A(\rho)+\frac{1}{2}\rho|\mathbf{v}|^2\right)(x,t)dx-\frac{1}{8\pi}\int|\nabla\Phi|^2(x,t)dx, \tag{3.5}$$

where as before,

$$A(\rho)=\frac{p(\rho)}{\gamma-1}. \tag{3.6}$$

Note that the energy $E(t)$ has both a positive and a negative part. This makes the stability analysis highly nontrivial, as noted in [30]. For a solution of (1.1) without shock waves, the total energy is conserved, i.e., $E(t)=E(0)$ $(t\geq 0)$(cf. [33]). For solutions with shock waves, the energy should be non-increasing in time, so that for all $t\geq 0$,

$$E(t)\leq E(0), \tag{3.7}$$

due to the entropy conditions, which are motivated by the second law of thermodynamics (cf. [18] and [32]). This will be proved in Theorem 5.1, below.

We consider axi-symmetric initial data, which takes the form

$$\begin{aligned} &\rho_0(x) = \rho(r,z), \\ &\mathbf{v}_0(x) = v_0^r(r,z)\mathbf{e}_r + v_0^\theta(r,z)\mathbf{e}_\theta + v_0^3(\rho,z)\mathbf{e}_3. \end{aligned} \tag{3.8}$$

Here $r = \sqrt{x_1^2 + x_2^2}$, $z = x_3$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ (as before), and

$$\mathbf{e}_r = (x_1/r, x_2/r, 0)^{\mathrm{T}},\ \mathbf{e}_\theta = (-x_2/r, x_1/r,\ 0)^{\mathrm{T}},\ \mathbf{e}_3 = (0,0,1)^{\mathrm{T}}. \tag{3.9}$$

We seek axi-symmetric solutions of the form

$$\rho(x,t) = \rho(r,z,t),$$

$$\mathbf{v}(x,t) = v^r(r,z,t)\mathbf{e}_r + v^\theta(r,z,t)\mathbf{e}_\theta + v^3(r,z,t)\mathbf{e}_3, \tag{3.10}$$

$$\Phi(x,t) = \Phi(r,z,t) = -B\rho(r,z,t), \tag{3.11}$$

We call a vector field $\mathbf{u}(x,t) = (u_1,u_2,u_3)(x)$ ($x \in \mathbb{R}^3$) axi-symmetric if it can be written in the form

$$\mathbf{u}(x) = u^r(r,z)\mathbf{e}_r + u^\theta(r,z)\mathbf{e}_\theta + u^3(\rho,z)\mathbf{e}_3.$$

For the velocity field $\mathbf{v} = (v_1,v_2,v_3)(x,t)$, we define the angular momentum $j(x,t)$ about the x_3-axis at (x,t) , $t \geq 0$, by

$$j(x,t) = x_1 v_2 - x_2 v_1. \tag{3.12}$$

For an axi-symmetric velocity field

$$\mathbf{v}(x,t) = v^r(r,z,t)\mathbf{e}_r + v^\theta(r,z,t)\mathbf{e}_\theta + v^3(\rho,z,t)\mathbf{e}_3, \tag{3.13}$$

$$v_1 = \frac{x_1}{r}v^r - \frac{x_2}{r}v^\theta,\ v_2 = \frac{x_2}{r}v^r + \frac{x_1}{r}v^\theta, v_3 = v^3, \tag{3.14}$$

so that

$$j(x,t) = rv^\theta(r,z,t). \tag{3.15}$$

In view of (3.13) and (3.15), we have

$$|\mathbf{v}|^2 = |v^r|^2 + \frac{j^2}{r^2} + |v^3|^2. \tag{3.16}$$

Therefore, the total energy at time t can be written as

$$\begin{aligned} E(\rho(t),\mathbf{v}(t)) = &\int A(\rho)(x,t)dx + \frac{1}{2}\int \frac{\rho j^2(x,t)}{r^2(x)}dx \\ &- \frac{1}{8\pi}\int |\nabla B\rho|^2(x,t)dx + \frac{1}{2}\int \rho(|v^r|^2 + |v^3|^2)(x,t)dx. \end{aligned} \tag{3.17}$$

There are two important conserved quantities for the Euler-Poisson equations (1.1); namely the total mass and the angular momentum. In order to describe these, we define D_t, the non-vacuum region at time $t \geq 0$ of the solution by

$$D_t = \{x \in \mathbb{R}^3 : \rho(x,t) > 0\}. \tag{3.18}$$

We will make the following physically reasonable assumptions A1)-A4) on weak solutions of the Cauchy problem (1.1) and (3.1):

A1) For any $t \geq 0$, there exists a measurable subset $G_t \subset D_t$ with $meas(D_t - G_t) = 0$ ($meas$ denotes the Lebsegue measure) such that, for any $x \in G_t$, there exists a unique (backwards) particle path $\xi(\tau, x, t)$ for $0 \leq \tau \leq t$ satisfying

$$\partial_\tau \xi(\tau, x, t) = \mathbf{v}(\xi(\tau, x, t), \tau), \ \xi(t, x, t) = x. \tag{3.19}$$

For $x \in G_t$, we write

$$\xi(0, x, t) = \xi_{-t}(x).$$

Also, for $x \in \mathbb{R}^3$ and $t \geq 0$, we denote the total mass at time t in the cylinder $\{y \in \mathbb{R}^3 : r(y) \leq r(x)\}$ by $m_{\rho(t)}(r(x))$, i.e.,

$$m_{\rho(t)}(r(x)) = \int_{r(y) \leq r(x)} \rho(y, t) dy. \tag{3.20}$$

For axi-symmetric motion, we assume

A2)

$$m_{\rho(t)}(r(x)) = m_{\rho_0}(r(\xi_{-t}(x))), \qquad \text{for } x \in G_t, t \geq 0. \tag{3.21}$$

Moreover, the angular momentum is conserved along the particle path:

A3)

$$j(x, t) = j(\xi_{-t}(x), 0), \qquad \text{for } x \in G_t, t \geq 0. \tag{3.22}$$

(Both (3.21) and (3.22) are shown in [33] if the solution has some regularity.)

Finally, for $L = j^2$, we need a technical assumption; namely,
A4)

$$\lim_{r \to 0+} \frac{L(m_{\rho(t)}(r) + m_{\tilde{\rho}}(r)) m_{\sigma(t)}(r)}{r^2} = 0, \tag{3.23}$$

for $t \geq 0$, where $\sigma(t) = \rho(t) - \tilde{\rho}$.

Remark 4. (3.23) can be understood as follows. For any $\rho \in W_M$, we have $\lim_{r \to 0+} m_\rho(r) = 0$. Therefore $\lim_{r \to 0+} L(m_{\rho(t)}(r) + m_{\tilde{\rho}}(r)) = L(0) = 0$, so if we define

$$\hat{\rho}(s, t) - \hat{\tilde{\rho}}(s) = \int_{-\infty}^{+\infty} (\rho(s, z, t) - \tilde{\rho}(s, z)) dz,$$

then if

$$\frac{m_{\sigma(t)}(r)}{r^2} = \frac{\int_0^r (2\pi s(\hat{\rho}(s, t) - \hat{\tilde{\rho}}(s)) ds}{r^2} \in L^\infty(0, \delta) \ for\ some\ \delta > 0, \tag{3.24}$$

(3.23)will hold. If $\hat{\rho}(\cdot, t) - \hat{\tilde{\rho}}(\cdot) \in L^\infty(0, \delta)$, then (3.24) holds. This can be assured by assuming that $\rho(r, z, t) - \tilde{\rho}(r, z) \in L^\infty((0, \delta) \times \mathbb{R} \times \mathbb{R}^+)$ and decays fast enough in the z direction. For example, when $\rho(x, t) - \tilde{\rho}(x)$ has compact support in $\mathbb{R}^3$ and $\rho(\cdot, t) - \tilde{\rho}(\cdot) \in L^\infty(\mathbb{R}^3)$, then (3.23) holds.

Now we make some assumptions on the initial data; namely, we assume that the initial data is such that the initial total mass and angular momentum are the same as those of the rotating star solution (those two quantities are conserved quantities). Therefore, we require

I_1)

$$\int \rho_0(x)dx = \int \tilde{\rho}(x)dx = M. \tag{3.25}$$

Moreover we assume

I_2) For the initial angular momentum $j(x,0) = rv_0^\theta(r,z) =: j_0(r,z)$ ($r = \sqrt{x_1^2 + x_2^2}$, $z = x_3$ for $x = (x_1, x_2, x_3)$, we assume $j(x,0)$ only depends on the total mass in the cylinder $\{y \in \mathbb{R}^3, r(y) \le r(x)\}$, i.e. ,

$$j(x,0) = j_0\left(m_{\rho_0}(r(x))\right). \tag{3.26}$$

Finally, we assume that the initial profile of the angular momentum per unit mass is the same as that of the rotating star solution, i. e.,

I_3)

$$j_0^2(m) = L(m), \qquad 0 \le m \le M, \tag{3.27}$$

where $L(m)$ is the profile of the square of the angular momentum of the rotating star defined in Section 2. ((3.26) implies that we require that $v_0^\theta(r,z)$ only depends on r.)
In order to state our stability result, we need some notation. Let λ be the number in Theorem 2.2, i.e.,

$$\begin{cases} A'(\tilde{\rho}(x)) + \int_{r(x)}^{\infty} L(m_{\tilde{\rho}}(s))s^{-3}ds - B\tilde{\rho}(x) = \lambda, \ x \in \Gamma, \\ \int_{r(x)}^{\infty} L(m_{\tilde{\rho}})(s))s^{-3}ds - B\tilde{\rho}(x) \ge \lambda, \qquad x \in \mathbb{R}^3 - \Gamma, \end{cases} \tag{3.28}$$

with A defined in (2.3), and Γ defined in (2.10).

For $\rho \in L^1 \cap L^\gamma$, we define,

$$d(\rho, \tilde{\rho}) = \int [A(\rho) - A(\tilde{\rho})] + (\rho - \tilde{\rho}) \int_{r(x)}^{\infty} \{\frac{L(m_{\tilde{\rho}}(s))}{s^3}ds - \lambda - B\tilde{\rho}\}dx. \tag{3.29}$$

Remark 5. For $x \in \Gamma$, in view of (3.6) and (3.28), we have,

$$\begin{aligned} &(A(\rho) - A(\tilde{\rho}))(x) + (\int_{r(x)}^{\infty} \frac{L(m_{\tilde{\rho}}(s))}{s^3}ds - \lambda - B\tilde{\rho}(x))(\rho - \tilde{\rho}) \\ &= (A(\rho) - A(\tilde{\rho}) - A'(\tilde{\rho})(\rho - \tilde{\rho}))(x) \\ &= \frac{p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho})(\rho - \tilde{\rho})}{\gamma - 1}(x) \ge 0. \end{aligned} \tag{3.30}$$

Thus, for $\rho \in W_M$,

$$d(\rho, \tilde{\rho}) \ge 0. \tag{3.31}$$

Moreover, $d(\rho, \tilde{\rho}) = 0$ if and only if $\rho = \tilde{\rho}$, and if $\gamma \le 2$,

$$d(\rho, \tilde{\rho}) \ge C||\rho - \tilde{\rho}||_2^2, \qquad \rho \in W_M. \tag{3.32}$$

We also define

$$d_1(\rho,\tilde{\rho}) = \frac{1}{2}\int \frac{\rho(x)L(m_\rho(r(x)) - \tilde{\rho}(x)L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx$$
$$-\int\int_{r(x)}^{\infty} s^{-3}L(m_{\tilde{\rho}}(s))ds(\rho(x)-\tilde{\rho}(x))dx, \tag{3.33}$$

for $\rho \in W_M$. We shall show later that $d_1 \geq 0$. Our main stability result in this paper is the following global-in-time stability theorem.

Theorem 3.1. *Let $\tilde{\rho}$ be a minimizer of the functional F in W_M, and assume that it is unique up to a vertical shift. Suppose $\gamma > 4/3$ and the above assumptions A1)-A4) and I_1)- I_3) hold. Moreover, assume that the angular momentum of the rotating star solution $\tilde{\rho}$ satisfies (2.6), (2.12) and (2.13). Let $(\rho,\mathbf{v},\Phi)(x,t)$ be an axi-symmetric weak solution of the Cauchy problem (1.1), (3.1) with $\rho(\cdot,t)\in L^1\cap L^\gamma$, $\rho|\mathbf{v}|^2(\cdot,t)\in L^1$ and $\nabla\Phi(\cdot,t) = -\nabla B\rho(\cdot,t)\in L^2$. If the total energy $E(t)$ (c.f. (3.5)) is non-increasing with respect to t, then for every $\epsilon > 0$, there exists a number $\delta > 0$ such that if*

$$d(\rho_0,\tilde{\rho}) + \frac{1}{8\pi}||\nabla B\rho_0 - \nabla B\tilde{\rho}||_2^2 + |d_1(\rho_0,\tilde{\rho})|$$
$$+\frac{1}{2}\int \rho_0(x)(|v_0^r|^2 + |v_0^3|^2)(x)dx < \delta, \tag{3.34}$$

then there is a vertical shift $a\mathbf{e_3}$ ($a\in\mathbb{R}$, $\mathbf{e_3} = (0,0,1)$) such that, for every $t > 0$

$$d(\rho(t),T^a\tilde{\rho}) + \frac{1}{8\pi}||\nabla B\rho(t) - BT^a\tilde{\rho}||_2^2 + |d_1(\rho(t),T^a\tilde{\rho})|$$
$$+\frac{1}{2}\int \rho(x,t)(|v^r(x,t)|^2 + |v^3(x,t)|^2)dx < \epsilon, \tag{3.35}$$

where $T^a\tilde{\rho}(x) =: \tilde{\rho}(x + a\mathbf{e_3})$.

Remark 6. The vertical shift $a\mathbf{e_3}$ appearing in the theorem is analogous to a similar phenomenon which appears in the study of stability of viscous traveling waves in conservation laws, whereby convergence is to a "shift" of the original traveling wave.

Remark 7. Without the uniqueness assumption for the minimizer of F in W_M, we can have the following type of stability result, as observed in [30] for the non-rotating star solutions. Suppose the assumptions in Theorem 3.1 hold. Let $\mathcal{S}_M$ be the set of all minimizers of F in W_M and $(\rho,\mathbf{v},\Phi)(x,t)$ be an axi-symmetric weak solution of the Cauchy problem (1.1), (3.1) with $\rho(\cdot,t)\in L^1\cap L^\gamma$, $\rho|\mathbf{v}|^2(\cdot,t)\in L^1$ and let $\nabla\Phi(\cdot,t) = -\nabla B\rho(\cdot,t)\in L^2$. If the total energy $E(t)$ is non-increasing with respect to t, then for every $\epsilon > 0$, there exists a number $\delta > 0$ such that if

$$\inf_{\tilde{\rho}\in\mathcal{S}_M}\left[d(\rho_0,\tilde{\rho}) + \frac{1}{8\pi}||\nabla B\rho_0 - \nabla B\tilde{\rho}||_2^2 + |d_1(\rho_0,\tilde{\rho})|\right]$$
$$+\frac{1}{2}\int \rho_0(x)(|v_0^r|^2 + |v_0^3|^2)(x)dx < \delta, \tag{3.36}$$

then for every $t > 0$

$$\inf_{\tilde{\rho}\in\mathcal{S}_M}\left[d(\rho(t),T^a\tilde{\rho})+\frac{1}{8\pi}||\nabla B\rho(t)-BT^a\tilde{\rho}||_2^2+|d_1(\rho(t),T^a\tilde{\rho})|\right]$$
$$+\frac{1}{2}\int\rho(x,t)(|v^r(x,t)|^2+|v^3(x,t)|^2)(x)dx<\epsilon. \tag{3.37}$$

The proof of this follows exactly along the same line as that for Theorem 3.1.

In order to prove Theorem 3.1, we need several lemmas. First we have

Lemma 3.1. *Suppose the angular momentum of the rotating star solutions satisfies (2.6), (2.12) and (2.13). For any $\rho(x)\in W_M$, if*

$$\lim_{r\to 0+}L(m_\rho(r)+m_{\tilde{\rho}}(r))m_\sigma(r)r^{-2}=0, \tag{3.38}$$

where $\sigma=\rho-\tilde{\rho}$, then

$$d_1(\rho,\tilde{\rho})\geq 0, \tag{3.39}$$

where d_1 is defined by (3.33).

Proof. First, we introduce some notation. For an axi-symmetric function $f(x)=f(r,z)$ ($r=\sqrt{x_1^2+x_2^2},\ z=x_3$ for $x=(x_1,x_2,x_3)$), we let

$$\hat{f}(r)=2\pi r\int_{-\infty}^{+\infty}f(r,z)dz, \tag{3.40}$$

$$m_f(r)=\int_{\{x:\sqrt{x_1^2+x_2^2}\leq r\}}f(x)dx=\int_0^r\hat{f}(s)ds, \tag{3.41}$$

so that

$$m_f'(r)=\hat{f}(r). \tag{3.42}$$

In order to show (3.39), we let

$$\sigma(x)=(\rho-\tilde{\rho})(x), \tag{3.43}$$

and for $0\leq\alpha\leq 1$, we define

$$Q(\alpha)=\frac{1}{2}\int\frac{(\tilde{\rho}+\alpha\sigma)(x)L(m_{\tilde{\rho}+\alpha\sigma}(r(x)))-\tilde{\rho}(x)L(m_{\tilde{\rho}}(r(x)))}{r^2(x)}dx$$
$$-\alpha\int\int_{r(x)}^{\infty}s^{-3}L(m_{\tilde{\rho}}(s))ds\sigma(x)dx. \tag{3.44}$$

Then

$$Q(0)=0,\ Q(1)=d_1(\rho,\ \tilde{\rho}). \tag{3.45}$$

Since

$$m_{\tilde{\rho}+\alpha\sigma}(r(x))=\int_0^{r(x)}2\pi s\int_{-\infty}^{+\infty}(\tilde{\rho}+\alpha\sigma)(s,z)dzds, \tag{3.46}$$

we have

$$\frac{d}{d\alpha}m_{\tilde{\rho}+\alpha\sigma}(r(x)) = \int_0^{r(x)} 2\pi s \int_{-\infty}^{+\infty} \sigma(s,z)dzds = m_\sigma(r(x)). \tag{3.47}$$

Therefore,

$$\begin{aligned} Q'(\alpha) = & \frac{1}{2}\int \frac{\sigma(x)L(m_{\tilde{\rho}+\alpha\sigma}(r(x)))}{r^2(x)}dx \\ & + \frac{1}{2}\int \frac{(\tilde{\rho}+\alpha\sigma)(x)L'(m_{\tilde{\rho}+\alpha\sigma}(r(x)))m_\sigma(r(x))}{r^2(x)}dx \\ & - \int\int_{r(x)}^{\infty} s^{-3}L(m_{\tilde{\rho}}(s))ds\sigma(x)dx, \end{aligned} \tag{3.48}$$

and in view of (3.42),

$$\frac{d}{dr}L(m_{\tilde{\rho}+\alpha\sigma}(r)) = L'(m_{\tilde{\rho}+\alpha\sigma}(r))(\ \hat{\tilde{\rho}}+\alpha\hat{\sigma})(r). \tag{3.49}$$

Therefore, by virtue of (3.49) and (3.42), we obtain

$$\begin{aligned} & \frac{1}{2}\int \frac{(\tilde{\rho}+\alpha\sigma)(x)L'(m_{\tilde{\rho}+\alpha\sigma}(r(x)))m_\sigma(r(x))}{r^2(x)}dx \\ = & \frac{1}{2}\int_0^{+\infty} (\ \hat{\tilde{\rho}}+\alpha\hat{\sigma})(r)L'(m_{\tilde{\rho}+\alpha\sigma}(r))m_\sigma(r)r^{-2}dr \\ = & \frac{1}{2}\int_0^{+\infty} \frac{d}{dr}[L(m_{\tilde{\rho}+\alpha\sigma}(r))]m_\sigma(r)r^{-2}dr. \end{aligned} \tag{3.50}$$

For $0 \le \alpha \le 1$, since (cf. (2.13)) $L'(m) \ge 0$, we have

$$L(m_{\tilde{\rho}+\alpha\sigma}(r)) \le L(m_{\tilde{\rho}+\rho}(r)). \tag{3.51}$$

This, together with (3.38), implies

$$\lim_{r\to 0+} L(m_{\tilde{\rho}+\alpha\sigma}(r))m_\sigma(r)r^{-2} = 0. \tag{3.52}$$

Moreover, since $m_\sigma(+\infty) = \int \sigma(x)dx = \int(\rho-\tilde{\rho})(x) = 0$ and

$$\lim_{r\to\infty} L(m_{\tilde{\rho}+\alpha\sigma}(r) = L(M),$$

we have

$$\lim_{r\to\infty} L(m_{\tilde{\rho}+\alpha\sigma}(r))m_\sigma(r)r^{-2} = 0. \tag{3.53}$$

It follows from (3.50), (3.52), (3.53) and integration by parts that

$$\begin{aligned} & \frac{1}{2}\int \frac{(\tilde{\rho}+\alpha\sigma)(x)L'(m_{\tilde{\rho}+\alpha\sigma}(r(x)))m_\sigma(r(x))}{r^2(x)}dx \\ = & -\frac{1}{2}\int_0^{+\infty} \hat{\sigma}(r)L(m_{\tilde{\rho}+\alpha\sigma}(r))m_\sigma(r)r^{-2}dr \\ & + \int_0^{+\infty} L(m_{\tilde{\rho}+\alpha\sigma}(r))m_\sigma(r)r^{-3}dr. \end{aligned} \tag{3.54}$$

Since
$$\int_0^{+\infty} \hat{\sigma}(r)L(m_{\tilde{\rho}+\alpha\sigma}(r))m_\sigma(r)r^{-2}dr = \int \frac{\sigma(x)L(m_{\tilde{\rho}+\alpha\sigma}(r(x)))}{r^2(x)}dx, \tag{3.55}$$
and
$$\int\int_{r(x)}^{\infty} s^{-3}L(m_{\tilde{\rho}}(s))ds\sigma(x)dx = \int_0^{+\infty} \hat{\sigma}(r)\int_r^{\infty} s^{-3}L(m_{\tilde{\rho}}(s))dsdr, \tag{3.56}$$
(3.48) and (3.54) imply
$$\begin{aligned} Q'(\alpha) = &\int_0^{+\infty} L(m_{\tilde{\rho}+\alpha\sigma}(r))m_\sigma(r)r^{-3}dr \\ &- \int_0^{+\infty} \hat{\sigma}(r)\int_r^{\infty} s^{-3}L(m_{\tilde{\rho}}(s))dsdr. \end{aligned} \tag{3.57}$$
Using (3.41), we have $m_\sigma(r) = \int_0^r \hat{\sigma}(s)ds$, so substituting this into the first term in (3.57) and interchanging the order of integration gives
$$\begin{aligned} &\int_0^{+\infty} L(m_{\tilde{\rho}+\alpha\sigma}(r))m_\sigma(r)r^{-3}dr \\ &= \int_0^{+\infty}\int_0^{r} r^{-3}L(m_{\tilde{\rho}+\alpha\sigma}(r))\hat{\sigma}(s)dsdr \\ &= \int_0^{+\infty}\hat{\sigma}(s)\int_s^{+\infty} r^{-3}L(m_{\tilde{\rho}+\alpha\sigma}(r))drds \\ &= \int_0^{+\infty}\hat{\sigma}(r)\int_r^{+\infty} s^{-3}L(m_{\tilde{\rho}+\alpha\sigma}(s))dsdr. \end{aligned} \tag{3.58}$$
Hence (3.57) and (3.58) yield
$$Q'(\alpha) = \int_0^{+\infty}\hat{\sigma}(r)\int_r^{\infty} s^{-3}(L(m_{\tilde{\rho}+\alpha\sigma}(s)) - L(m_{\tilde{\rho}}(s)))dsdr, \tag{3.59}$$
and therefore
$$Q(0) = Q'(0) = 0. \tag{3.60}$$
Differentiating (3.59) again, we obtain
$$\frac{d^2Q(\alpha)}{d\alpha^2} = \alpha\int_0^{+\infty}\hat{\sigma}(r)\int_r^{\infty} s^{-3}L'(m_{\tilde{\rho}+\alpha\sigma}(s))m_\sigma(s)dsdr, \tag{3.61}$$
and interchanging the order of integration gives
$$\frac{d^2Q(\alpha)}{d\alpha^2} = \alpha\int_0^{+\infty} s^{-3}\int_0^{s}\hat{\sigma}(r)drL'(m_{\tilde{\rho}+\alpha\sigma}(s))m_\sigma(s)ds. \tag{3.62}$$
Noting that $\int_0^s \hat{\sigma}(r)dr = m_\sigma(s)$, we obtain
$$\frac{d^2Q(\alpha)}{d\alpha^2} = \alpha\int_0^{+\infty} s^{-3}L'(m_{\tilde{\rho}+\alpha\sigma}(s))(m_\sigma(s))^2ds. \tag{3.63}$$

Therefore, if $L'(m) \geq 0$ for $0 \leq m \leq M$, then

$$\frac{d^2Q(\alpha)}{d\alpha^2} \geq 0, \ \ for \ 0 \leq \alpha \leq 1. \tag{3.64}$$

This, together with (3.60)and (3.45), yields $d_1(\rho, \tilde{\rho}) = Q(1) \geq 0$. □

Lemma 3.2. *Let $(\rho, \mathbf{v})$ be a solution of the Cauchy problem (1.1), (3.1) as stated in Theorem 3.1, then*

$$\begin{aligned}
&E(\rho, \mathbf{v})(t) - F(\tilde{\rho}) \\
&= d(\rho(t), \tilde{\rho}) + d_1(\rho(t), \tilde{\rho}) - \frac{1}{8\pi}||\nabla(B\rho(\cdot, t) - B\tilde{\rho})||_2^2 \\
&+ \frac{1}{2}\int \rho(|v^r|^2 + |v^3|^2)(x, t)dx.
\end{aligned} \tag{3.65}$$

Proof. From A1)-A3), for any $x \in G_t$ we have

$$j^2(x, t) = j_0^2(\xi_{-t}(x)), \tag{3.66}$$

(see (3.26)). In view of (3.22) and (3.27),

$$j^2(x, t) = j_0^2(\xi_{-t}(x)) = L(m_{\rho_0}(r(\xi_{-t}(x))), \tag{3.67}$$

for $x \in G_t$. This, together with (3.21), yields

$$j^2(x, t) = L(m_{\rho(t)}(r(x))), \qquad x \in G_t. \tag{3.68}$$

Therefore, by (3.17), we have

$$\begin{aligned}
E(\rho(t), \mathbf{v}(t)) = &\int A(\rho)(x, t)dx + \frac{1}{2}\int \frac{\rho(x, t)L(m_{\rho(t)}(r(x))}{r^2(x)}dx \\
&- \frac{1}{8\pi}\int |\nabla B\rho|^2(x, t)dx + \frac{1}{2}\int \rho(|v^r|^2 + |v^3|^2)(x, t)dx.
\end{aligned} \tag{3.69}$$

Here we have used the fact that

$$\int \frac{\rho(x, t)L(m_{\rho(t)}(r(x))}{r^2(x)}dx = \int_{G_t} \frac{\rho(x, t)L(m_{\rho(t)}(r(x))}{r^2(x)}dx,$$

which holds because $D_t = \{x \in R^3 : \rho(x, t) > 0\}$, $G_t \subset D_T$ and $meas(D_t - G_t) = 0$. It follows from (2.5) and (3.69) that

$$\begin{aligned}
&E(\rho, \mathbf{v})(t) - F(\tilde{\rho}) \\
&= \int (A(\rho)(x, t) - A(\tilde{\rho})(x))dx \\
&+ \frac{1}{2}\int \frac{\rho(x, t)L(m_{\rho(t)}(r(x)) - \tilde{\rho}(x)L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx \\
&- \frac{1}{8\pi}(||\nabla B\rho(x, t)||^2 - ||\nabla B\tilde{\rho}||_2^2 \\
&+ \frac{1}{2}\int \rho(|v^r|^2 + |v^3|^2)(x, t)dx.
\end{aligned} \tag{3.70}$$

On the other hand,

$$
\begin{aligned}
& -\frac{1}{8\pi}(||\nabla B\rho(\cdot,t)||_2^2 - ||\nabla B\tilde{\rho}||_2^2) \\
&= -\frac{1}{8\pi}||\nabla(B\rho(\cdot,t) - \nabla B\tilde{\rho})||_2^2 - \frac{1}{4\pi}\int \nabla B\tilde{\rho}(x)\cdot(\nabla B\rho(x,t) - \nabla B\tilde{\rho}(x))dx. \qquad (3.71)
\end{aligned}
$$

Noting that $\Delta(B\rho - B\tilde{\rho}) = -4\pi(\rho - \tilde{\rho})$, and integrating by parts (this is legitimate, cf. [29]) gives,

$$
\begin{aligned}
& -\frac{1}{4\pi}\int \nabla B\tilde{\rho}(x)\cdot(\nabla B\rho(x,t) - \nabla B\tilde{\rho}(x))dx \\
&= \frac{1}{4\pi}\int B\tilde{\rho}(x)(\Delta B\rho(x,t) - \Delta B\tilde{\rho}(x))dx \\
&= \int B\tilde{\rho}(x)(B\rho(x,t) - B\tilde{\rho}(x))dx. \qquad (3.72)
\end{aligned}
$$

By (3.70)-(3.72), and noting (3.33), we have

$$
\begin{aligned}
& E(\rho,\mathbf{v})(t) - F(\tilde{\rho}) \\
&= \int \left(A(\rho) - A(\tilde{\rho}) + (\rho - \tilde{\rho})\{\int_{r(x)}^{\infty} \frac{L(m_{\tilde{\rho}}(s))}{s^3}ds - B\tilde{\rho}\} \right) dx \\
&+ d_1(\rho(t),\tilde{\rho}) - \frac{1}{8\pi}(||\nabla(B\rho(x,t) - B\tilde{\rho})||_2^2 \\
&+ \frac{1}{2}\int \rho(|v^r|^2 + |v^3|^2)(x,t)dx. \qquad (3.73)
\end{aligned}
$$

Since $\rho(\cdot,t) \in W_M$, $\int \rho(x,t)dx = \int \tilde{\rho}(x)dx = M$. Thus $\int \lambda(\rho(x,t) - \tilde{\rho}(x))dx = 0$. Therefore, the first term in (3.73) is the same as $d(\rho(t),\tilde{\rho})$ defined by (3.29). This completes the proof of the lemma. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Assume the theorem is false. Then there exist $\epsilon_0 > 0$, $t_n > 0$ and initial data $\rho_n(x,0) \in W_M$ and $\mathbf{v}_n(x,0)$ such that for all $n \in \mathbb{N}$,

$$
\begin{aligned}
& d(\rho_n(0),\tilde{\rho}) + d_1(\rho_0,\tilde{\rho}) + \frac{1}{8\pi}||\nabla B\rho_n(0) - \nabla B\tilde{\rho}||_2^2 \\
&+ \frac{1}{2}\int \rho_n(x,0)(|v_n^r(x,0)|^2 + |v_n^3(x,0)|^2)(x)dx < \frac{1}{n}, \qquad (3.74)
\end{aligned}
$$

but for any $a \in \mathbb{R}$,

$$
\begin{aligned}
& d(\rho_n(t_n),T^a\tilde{\rho}) + d_1(\rho(t),T^a\tilde{\rho}) + \frac{1}{8\pi}||\nabla B\rho_n(t_n) - \nabla BT^a\tilde{\rho}||_2^2 \\
&+ \frac{1}{2}\int \rho_n(x,t_n)(|v_n^r(x,t_n)|^2 + |v_n^3(x,t_n)|^2)(x)dx \geq \epsilon_0. \qquad (3.75)
\end{aligned}
$$

By (3.65) and (3.74), we have

$$\lim_{n\to\infty} E(\rho_n(0), \mathbf{v}_n(0)) = F(\tilde{\rho}). \tag{3.76}$$

Since $E(\rho_n(t), \mathbf{v}_n(t))$ is non-increasing in time,

$$\lim_{n\to\infty} \sup F(\rho_n(t_n)) \le \lim_{n\to\infty} E(\rho_n(t_n), \mathbf{v}_n(t_n)) \le \lim_{n\to\infty} E(\rho_n(0), \mathbf{v}_n(0)) = F(\tilde{\rho}). \tag{3.77}$$

(The first inequality holds because we have, similar to (3.71),

$$E(\rho, \mathbf{v})(t) - F(\rho(t)) = \frac{1}{2}\int \rho(|v^r|^2 + |v^3|^2)(x,t)dx \ge 0, \ t \ge 0.)$$

Therefore $\{\rho_n(\cdot, t_n)\} \subset W_M$ is a minimizing sequence for the functional F. We apply Theorem 2.3 to conclude that there exists a sequence $\{a_n\} \subset \mathbb{R}$ such that up to a subsequence,

$$||\nabla(B\rho_n(t_n) - BT^{a_n}\tilde{\rho})||_2 \to 0, \tag{3.78}$$

as $n \to \infty$; this is where we use the assumption that the minimizer is unique up to a vertical shift. Note also that for any $\rho \in W_M$ and $a \in \mathbb{R}$,

$$\begin{cases} ||\nabla B(T^a\rho) - \nabla B\tilde{\rho}||_2 = ||\nabla B(\rho) - \nabla BT^{-a}\tilde{\rho}||_2, \\ d(T^a\rho, \tilde{\rho}) = d(\rho, T^{-a}\tilde{\rho}), \ \textit{and} \ d_1(T^a\rho, \tilde{\rho}) = d_1(\rho, T^{-a}\tilde{\rho}). \end{cases} \tag{3.79}$$

Thus, by (3.65), the fact that the energy is non-increasing in time, and $F(T^a\rho) = F(\rho)$, we have for any $\rho \in W_M$ and $a \in \mathbb{R}$,

$$\begin{aligned} & E(\rho_n(t_n), \mathbf{v}_n(t_n)) - F(T^{a_n}\tilde{\rho}) \\ & = d(\rho_n(t_n), T^{a_n}\tilde{\rho}) + d_1(\rho(t_n), T^{a_n}\tilde{\rho}) \\ & - \frac{1}{8\pi}||\nabla(B\rho_n(t_n) - BT^{a_n}\tilde{\rho})||_2^2 \\ & + \frac{1}{2}\int \rho_n(|v_n^r|^2 + |v_n^3|^2)(x, t_n)dx \\ & \le E(\rho_n(0), \mathbf{v}_n(0)) - F(T^{a_n}\tilde{\rho}) \\ & = E(\rho_n(0), \mathbf{v}_n(0)) - F(\tilde{\rho}) \to 0, \end{aligned} \tag{3.80}$$

as $n \to \infty$. Since

$$||\nabla B\rho_n(t_n) - \nabla BT^{a_n}\tilde{\rho}||_2 \to 0,$$

as $n \to \infty$, $d(\rho_n(t_n), \tilde{\rho}) \ge 0$ (cf. (3.31)) and $d_1(\rho(t_n), \tilde{\rho}) \ge 0$ (cf. A4) and (3.37)), we have

$$\begin{aligned} & d(\rho_n(t_n), T^{a_n}\tilde{\rho}) + d_1(\rho(t_n), T^{a_n}\tilde{\rho}) \\ & + \frac{1}{8\pi}||\nabla(B\rho_n(t_n) - T^{a_n}B\tilde{\rho})||_2^2 \\ & + \frac{1}{2}\int \rho_n(|v_n^r|^2 + |v_n^3|^2)(x, t_n)dx \to 0, \end{aligned} \tag{3.81}$$

as $n \to \infty$. This contradicts (3.75), and completes the proof.

4 Stability of General Entropy Solutions

In this section, we shall obtain a stability theorem for general entropy weak solutions. We begin with the definition of entropy weak solution.

Definition 4.1. A weak solution (defined in Section 3) on $\mathbb{R}^3 \times [0,T]$ is called an *entropy weak solution* of (1.1) if it satisfies the following "entropy inequality":

$$\partial_t \eta + \sum_{j=1}^{3} \partial_{x_j} q_j + \rho \sum_{j=1}^{3} \eta_{m_j} \Phi_{x_j} \leq 0, \tag{4.1}$$

in the sense of distributions; i.e.,

$$\int_0^T \int_{\mathbb{R}^3} \left(\eta \beta_t + \mathbf{q} \cdot \nabla \beta - \rho \sum_{j=1}^{3} \eta_{m_j} \Phi_{x_j} \beta \right) dxdt + \int_{\mathbb{R}^3} \beta(x,0)\eta(x,0)dx \geq 0, \tag{4.2}$$

for any nonnegative Lipschitz continuous test function β with compact support in $[0,T) \times \mathbb{R}^3$. Here the "entropy" function η and "entropy flux" functions q_j and $\mathbf{q}$, are defined by

$$\begin{cases} \eta = \frac{|\mathbf{m}|^2}{2\rho} + \rho \int_0^\rho \frac{p'(s)}{s^2} ds = \frac{|\mathbf{m}|^2}{2\rho} + \frac{\rho^\gamma}{\gamma - 1}, \\ q_j = \frac{|\mathbf{m}|^2 m_j}{2\rho^2} + m_j \int_0^\rho \frac{p'(s)}{s} ds = \frac{|\mathbf{m}|^2}{2\rho} + \frac{\gamma \rho^\gamma}{\gamma - 1} \quad (j = 1,2,3), \\ \mathbf{q} = (q_1, \ q_2, \ q_3). \end{cases} \tag{4.3}$$

Remark 8. The inequality (4.1) is motivated by the second law of thermodynamics ([18]), and plays an important role in shock wave theory ([32]). For smooth solutions, the inequality in (4.1) can be replaced by equality.

For a general entropy weak solution, our stability result is given by the following theorem:

Theorem 4.1. *Suppose* $1 < \gamma \leq 2$. *Let* $(\rho, \mathbf{m}, \Phi)(x,t)$ *(*$t \in [0,T]$, $x \in \mathbb{R}^3$*) with* $(\rho, \mathbf{m}) \in L^\infty(\mathbb{R}^3 \times [0,T])$*, be a weak solution of (1.1) satisfying the entropy condition (4.1) and let* $(\bar{\rho}, \bar{\mathbf{m}}, \bar{\Phi})(x,t)$*,* $t \in [0,T]$*,* $x \in \mathbb{R}^3$ *be any solution of (1.1) satisfying* $(\bar{\rho}, \bar{\mathbf{m}}) \in W^{1,\infty}_{loc}(\mathbb{R}^3 \times [0,T])$*. Assume*

$$Z(T) =: \sup_{0 \leq t \leq T} (||\rho(\cdot,t)||_\infty (||\rho(\cdot,t)||_\infty + ||\bar{\rho}(\cdot,t)||_\infty)^{2-\gamma} (VolS(t))^{2/3} + ||\nabla_x \bar{\mathbf{v}}(\cdot,t)||_\infty) < +\infty, \tag{4.4}$$

and

$$\frac{\mathbf{m}}{\rho}, \frac{\bar{\mathbf{m}}}{\bar{\rho}} \in L^\infty(\mathbb{R}^3 \times [0,T]). \tag{4.5}$$

where $S(t) = Supp|\rho - \bar{\rho}|(\cdot,t)$*. Then there is a constant* $C(T)$ *depending on* T *and* $Z(T)$ *such that*

$$Y(t) \leq C(T)Y(0), \qquad 0 \leq t \leq T, \tag{4.6}$$

where

$$Y(t) = D(\rho, \bar{\rho})(t) + ||\sqrt{\rho}(\nabla\Phi - \nabla\bar{\Phi})||_2^2(t) + \int \rho(x,t)|\mathbf{v} - \bar{\mathbf{v}}|^2(x,t)dx,$$

$$\Phi = -B\rho, \bar{\Phi} = -B\bar{\rho},$$

and

$$D(\rho,\ \bar{\rho}) = \int \frac{p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})(\rho - \bar{\rho})}{\gamma - 1} dx.$$

Remark 9. The function $(\bar{\rho}, \bar{\mathbf{m}}, \bar{\Phi})$ in the theorem could be, but is not necessarily, a rotating star solution.

Remark 10. For $1 < \gamma \le 2$, it is easy to see

$$D(\rho,\ \bar{\rho}) \ge C||\rho - \bar{\rho}||_2^2,$$

for some constant $C > 0$ if $\rho, \bar{\rho} \in L^\infty(\mathbb{R}^3 \times [0, T])$.

Proof of Theorem 4.1
Letting $U = (\rho, \mathbf{m})^{\mathrm{T}}$ with $\mathbf{m} = (m_1, m_2, m_3) = \rho\mathbf{v}$ and $\bar{U} = (\bar{\rho}, \bar{\mathbf{m}})^{\mathrm{T}}$, we can write system (1.1) as

$$\begin{cases} U_t + \sum_{j=1}^3 F_j(U)_{x_j} = -\rho\nabla\Phi, \\ \Delta\Phi = 4\pi\rho. \end{cases} \tag{4.7}$$

Here the flux functions $F_j(U)$ are given by

$$\begin{cases} F_1(U) = \left(m_1, p(\rho) + \frac{m_1^2}{\rho},\ \frac{m_1 m_2}{\rho},\ \frac{m_1 m_3}{\rho}\right)^{\mathrm{T}}, \\ F_2(U) = \left(m_2, \frac{m_1 m_2}{\rho}, p(\rho) + \frac{m_2^2}{\rho},\ \frac{m_2 m_3}{\rho}\right)^{\mathrm{T}}, \\ F_3(U) = \left(m_3, \frac{m_1 m_3}{\rho}, \frac{m_2 m_3}{\rho},\ p(\rho) + \frac{m_3^2}{\rho}\right)^{\mathrm{T}}. \end{cases} \tag{4.8}$$

The entropy and entropy fluxes η and $\mathbf{q}$ are as in (4.3) and satisfy

$$\nabla q_j(U) = \nabla\eta(U)\nabla F_j(U), \qquad j = 1,\ 2, 3, \tag{4.9}$$

as is easily verifiable. Since U is an entropy weak solution

$$\partial_t\eta(U) + \sum_{j=1}^3 \partial_{x_j} q_j(U) + \rho\sum_{j=1}^3 \eta_{m_j}(U)\Phi_{x_j} \le 0, \tag{4.10}$$

in the sense of distributions. Because $\bar{U} \in W_{loc}^{1,\,\infty}$ is a weak solution of (1.1), we have

$$\partial_t\eta(\bar{U}) + \sum_{j=1}^3 \partial_{x_j} q_j(\bar{U}) + \rho\sum_{j=1}^3 \eta_{m_j}(\bar{U})\bar{\Phi}_{x_j} = 0. \tag{4.11}$$

We define the relative entropy-entropy flux pairs by

$$\begin{cases} \eta^*(U,\bar{U}) = \eta(U) - \eta(\bar{U}) - \nabla\eta(\bar{U})(U - \bar{U}), \\ q_j^*(U,\bar{U}) = q_j(U) - q_j(\bar{U}) - \nabla\eta(\bar{U})(F_j(U) - F_j(\bar{U}))\ (j = 1, 2, 3). \end{cases} \tag{4.12}$$

Using (4.10) and (4.11) gives

$$\begin{aligned} &\partial_t\eta^* + \sum_{j=1}^{3}\partial_{x_j}q_j^* \\ &= (\partial_t\eta(U) + \sum_{j=1}^{3}\partial_{x_j}q_j(U)) - (\partial_t\eta(\bar{U}) + \sum_{j=1}^{3}\partial_{x_j}q_j(\bar{U})) \\ &- \nabla^2\eta(\bar{U})\{(\bar{U}_t, U - \bar{U}) + (\sum_{j=1}^{3}\partial_{x_j}\bar{U}, F_j(U) - F_j(\bar{U}))\} \\ &- \nabla\eta(\bar{U})\{(U - \bar{U})_t + \sum_{j=1}^{3}\partial_{x_j}(F_j(U) - F_j(\bar{U}))\} \\ &\le (\nabla\eta(U) - \nabla\eta(\bar{U}))R - \nabla^2\eta(\bar{U})(\bar{R}, U - \bar{U}) \\ &- \nabla^2\eta(\bar{U})\sum_{j=1}^{3}\left(\partial_{x_j}\bar{U},\ F_j(U) - F_j(\bar{U}) - F_j'(\bar{U})(U - \bar{U})\right), \end{aligned} \tag{4.13}$$

in the sense of distributions, where

$$R = (0, -\rho\nabla\Phi)^{\mathrm{T}},\ \text{and}\ \bar{R} = (0, -\bar{\rho}\nabla\bar{\Phi})^{\mathrm{T}}. \tag{4.14}$$

It is easy to check that

$$\begin{aligned} &(\nabla\eta(U) - \nabla\eta(\bar{U}))R - \nabla^2\eta(\bar{U})(\bar{R}, U - \bar{U}) \\ &= -\rho(\mathbf{v} - \bar{\mathbf{v}}) \cdot (\nabla\Phi - \nabla\bar{\Phi}), \end{aligned} \tag{4.15}$$

so that

$$\begin{aligned} &\partial_t\eta^* + \sum_{j=1}^{3}\partial_{x_j}q_j^* \\ &\le -\rho(\mathbf{v} - \bar{\mathbf{v}}) \cdot (\nabla\Phi - \nabla\bar{\Phi}) \\ &- \nabla^2\eta(\bar{U})\sum_{j=1}^{3}\left(\partial_{x_j}\bar{U},\ F_j(U) - F_j(\bar{U}) - F_j'(\bar{U})(U - \bar{U})\right), \end{aligned} \tag{4.16}$$

in the sense of distributions. That is, for any nonnegative, Lipschitz continuous test function

ψ on $\mathbb{R}^3 \times [0,T)$, with compact support, we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \left(\partial_t \psi \eta^* + \sum_{j=1}^3 \partial_j \psi q_j^* \right) dxdt + \int_{\mathbb{R}^3} \psi(x,0)\eta^*(x,0)dx \\
&\geq \int_0^T \int_{\mathbb{R}^3} \psi\rho(\mathbf{v}-\bar{\mathbf{v}}) \cdot (\nabla\Phi - \nabla\bar{\Phi})dxdt \\
&+ \int_0^T \int_{\mathbb{R}^3} \psi \nabla^2\eta(\bar{U}) \sum_{j=1}^3 \left(\partial_{x_j}\bar{U},\ F_j(U) - F_j(\bar{U}) - F_j'(\bar{U})(U-\bar{U}) \right) dxdt. \qquad (4.17)
\end{aligned}
$$

A calculation gives

$$
\nabla^2\eta(U) = \begin{pmatrix} \frac{m^2}{\rho^3} + \frac{p''(\rho)}{\gamma-1} & -\frac{m_1}{\rho^2} & -\frac{m_2}{\rho^2} & -\frac{m_3}{\rho^2} \\ -\frac{m_1}{\rho^2} & \frac{1}{\rho} & 0 & 0 \\ -\frac{m_2}{\rho^2} & 0 & \frac{1}{\rho} & 0 \\ -\frac{m_3}{\rho^2} & 0 & 0 & \frac{1}{\rho} \end{pmatrix}, \qquad (4.18)
$$

and also

$$
\eta^* = \frac{p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})(\rho - \bar{\rho})}{\gamma - 1} + \frac{1}{2}\rho|\vec{v} - \bar{\vec{v}}|^2. \qquad (4.19)
$$

So, for $1 < \gamma \leq 2$,

$$
\eta^* \geq c_1(||\rho(\cdot,t)||_\infty + ||\bar{\rho}(\cdot,t)||_\infty)^{\gamma-2}(\rho-\bar{\rho})^2 + \frac{1}{2}\rho|\mathbf{v}-\bar{\mathbf{v}}|^2 \geq 0, \qquad (4.20)
$$

for some positive constant c_1.

A further calculation yields, using (4.18),

$$
\begin{aligned}
&\nabla^2\eta(\bar{U}) \sum_{j=1}^3 \left(\partial_{x_j}\bar{U},\ F_j(U) - F_j(\bar{U}) - F_j'(\bar{U})(U-\bar{U}) \right) \\
&= \{p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})(\rho-\bar{\rho})\} \sum_{j=1}^3 \partial_j \bar{v}_j \\
&+ \frac{1}{2} \sum_{i,j=1}^3 \rho(v_i - \bar{v}_i)(v_j - \bar{v}_j)(\partial_j \bar{v}_i + \partial_i \bar{v}_j). \qquad (4.21)
\end{aligned}
$$

Here and in the following, we use the notation:

$$
\partial_j = \frac{\partial}{\partial x_j}, \qquad j = 1,\ 2,\ 3.
$$

Therefore, by (4.19) and (4.21), we have

$$
\begin{aligned}
&|\nabla^2\eta(\bar{U}) \sum_{j=1}^3 \left(\partial_{x_j}\bar{U},\ F_j(U) - F_j(\bar{U}) - F_j'(\bar{U})(U-\bar{U}) \right)|(x,t) \\
&\leq C||\nabla_x \bar{\mathbf{v}}(\cdot,t)||_\infty \eta^*(x,t), \qquad (4.22)
\end{aligned}
$$

for $x \in \mathbb{R}^3$, $t \in [0,T)$ and some constant $C > 0$. Thus, (4.17)-(4.22) yield

$$\int_0^T \int_{\mathbb{R}^3} \left(\partial_t \psi \eta^* + \sum_{j=1}^3 \partial_j \psi q_j^* \right) dxdt + \int_{\mathbb{R}^3} \psi(x,0)\eta^*(x,0)dx$$
$$\geq \int_0^T \int_{\mathbb{R}^3} \psi\rho(\mathbf{v} - \bar{\mathbf{v}}) \cdot (\nabla\Phi - \nabla\bar{\Phi})dxdt$$
$$- C \sup_{0\leq t\leq T} ||\nabla_x \bar{\mathbf{v}}(\cdot,t)||_\infty \int_0^T \int_{\mathbb{R}^3} \psi\eta^*(x,t)dxdt. \tag{4.23}$$

Using (4.5), it is easy to see that there exists a positive constant Λ, which may depend on T, such that

$$(\sum_{j=1}^3 |q_j^*|^2)^{1/2}(x,t) \leq \Lambda\eta^*(x,t), \qquad (x,t) \in \mathbb{R}^3 \times [0,T]. \tag{4.24}$$

For fixed $L > 0$, $t \in (0,T)$ and small $\epsilon > 0$, we consider the test function $\psi(x,\tau) = \varsigma(x,\tau)\vartheta(\tau)$ defined by

$$\vartheta(\tau) = \begin{cases} 1, & 0 \leq \tau < t \\ \frac{1}{\epsilon}(t-\tau) + 1, & t \leq \tau < t+\epsilon \\ 0, & t+\epsilon \leq \tau < T, \end{cases} \tag{4.25}$$

$$\varsigma(x,\tau) = \begin{cases} 1, & (x,\tau) \in R_1 \\ \frac{1}{\epsilon}[L + \Lambda(t-\tau) - |x|] + 1, & (x,\tau) \in R_2 \\ 0, & (x,\tau) \in R_3, \end{cases} \tag{4.26}$$

where

$$R_1 = \{(x,\tau) : 0 \leq \tau < T,\ 0 \leq |x| < L + \Lambda(t-\tau)\},$$
$$R_2 = \{(x,\tau) : 0 \leq \tau < T, L + \Lambda(t-\tau) \leq |x| < L + \Lambda(t-\tau) + \epsilon\},$$
$$R_3 = \{(x,\tau) : 0 \leq \tau < T,\ |x| > L + \Lambda(t-\tau) + \epsilon\},$$

and Λ is the constant given in (4.24). Substituting this in (4.23), a straightforward calculation yields,

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} \int_{|x|<L} \eta^*(x,\tau)dxd\tau$$
$$\leq \int_{|x|<L+\Lambda t} \eta^*(x,0)dx$$
$$- \frac{1}{\epsilon} \int_0^t \int_{L+\Lambda(t-\tau)\leq|x|<L+\Lambda(t-\tau)+\epsilon} \left\{ \Lambda\eta^* + \sum_{j=1}^3 \frac{x_j}{|x|} q_j^* \right\} dxd\tau$$
$$- \int_0^t \int_{|x|<L+\Lambda(t-\tau)} \rho(\mathbf{v} - \bar{\mathbf{v}}) \cdot (\nabla\Phi - \nabla\bar{\Phi})dxd\tau$$
$$+ C \sup_{0\leq\tau\leq T} ||\nabla_x \bar{\mathbf{v}}(\cdot,\tau)||_\infty \int_0^t \int_{|x|<L+\Lambda(t-\tau)} \eta^*(x,\tau)dxd\tau + O(\epsilon). \tag{4.27}$$

The second term on the right-had side of (4.27) is negative in view of (4.24), together with Cauchy-Schwartz inequality. Letting $\epsilon \to 0^+$ in (4.27) gives

$$\begin{aligned}
&\int_{|x|<L} \eta^*(x,t)dx \\
&\leq \int_{|x|<L+\Lambda t} \eta^*(x,0)dx \\
&- \int_0^t \int_{|x|<L+\Lambda(t-\tau)} \rho(\mathbf{v}-\bar{\mathbf{v}}) \cdot (\nabla\Phi - \nabla\bar{\Phi})dxd\tau \\
&+ C \sup_{0\leq\tau\leq T} ||\nabla_x \bar{\mathbf{v}}(\cdot,\tau)||_\infty \int_0^t \int_{|x|<L+\Lambda(t-\tau)} \eta^*(x,\tau)dxd\tau. \qquad (4.28)
\end{aligned}$$

We now let $L \to +\infty$ in (4.28) to get

$$\begin{aligned}
&\int \eta^*(x,t)dx \\
&\leq \int \eta^*(x,0)dx \\
&- \int_0^t \int \rho(\mathbf{v}-\bar{\mathbf{v}}) \cdot (\nabla\Phi - \nabla\bar{\Phi})dxd\tau \\
&+ C \sup_{0\leq\tau\leq T} ||\nabla_x \bar{\mathbf{v}}(\cdot,\tau)||_\infty \int_0^t \int \eta^*(x,\tau)dxd\tau. \qquad (4.29)
\end{aligned}$$

The second term on the right hand side can be estimated as follows. By Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&|\int \rho(\mathbf{v}-\bar{\mathbf{v}}) \cdot (\nabla\Phi - \bar{\nabla}\Phi)(x,\tau)dx \\
&\leq \frac{1}{2}\int \rho|\mathbf{v}-\bar{\mathbf{v}}|^2(x,\tau)dx + \frac{1}{2}\int \rho|\nabla\Phi - \bar{\nabla}\Phi|^2(x,\tau)dx. \qquad (4.30)
\end{aligned}$$

Applying Lemma 2.3, we obtain

$$\begin{aligned}
&\int \rho|\nabla\Phi - \bar{\nabla}\Phi|^2(x,t)dx \\
&\leq ||\rho(\cdot,\tau)||_\infty ||\nabla(\Phi-\bar{\Phi})(\cdot,\tau||_2^2 \\
&\leq C||\rho(\cdot,\tau)||_\infty \left(\int |\rho-\bar{\rho}|^{4/3}(x,t)dx\right)\left(\int |\rho-\bar{\rho}|^{4/3}(x,\tau)dx\right)^{2/3} \\
&= C||\rho(\cdot,t)||_\infty \left(\int_{S(\tau)} |\rho-\bar{\rho}|^{4/3}(x,t)dx\right)\left(\int_{S(\tau)} |\rho-\bar{\rho}|^{4/3}(x,\tau)dx\right)^{2/3}, \qquad (4.31)
\end{aligned}$$

where

$$S(\tau) = supp|\rho-\bar{\rho}|(\cdot,\tau).$$

It follows from Hölder's inequality that

$$\int_{S(t)} |\rho-\bar{\rho}|^{4/3}(x,t)dx \leq \left(\int_{S(\tau)} |\rho-\bar{\rho}|^2(x,\tau)dx\right)^{2/3} (volS(\tau))^{1/3}, \qquad (4.32)$$

and

$$\left(\int_{S(t)} |\rho-\bar{\rho}|(x,\tau)dx\right)^{2/3} \leq \left(\int_{S(t)} |\rho-\bar{\rho}|^2(x,\tau)dx\right)^{1/3} (volS(\tau))^{1/3}. \qquad (4.33)$$

Then using (4.31)-(4.33) we obtain

$$\int \rho|\nabla\Phi-\bar{\nabla}\Phi|^2(x,\tau)dx \leq C||\rho(\cdot,\tau)||_\infty(||\rho(\cdot,\tau)||_\infty+||\bar{\rho}(\cdot,\tau)||_\infty)^{2-\gamma}||(\rho-\bar{\rho})(\cdot,\tau)||_2^2(VolS(t)))^{2/3}. \qquad (4.34)$$

In view of (4.20), (4.29), (4.30) and (4.34), we have

$$\int \eta^*(x,t)dx \leq \int \eta^*(x,0)dx + CZ(T)\int_0^t \int \eta^*(x,\tau)dxd\tau, \qquad (4.35)$$

for $0 \leq t \leq T$, where

$$Z(T) =: sup_{0\leq t\leq T}(||\rho(\cdot,t)||_\infty(||\rho(\cdot,t)||_\infty + ||\bar{\rho}(\cdot,t)||_\infty)^{2-\gamma}(VolS(t))^{2/3} + ||\nabla_x\bar{\mathbf{v}}(\cdot,t)||_\infty).$$

Then (4.6) follows from Gronwall's inequality applied to (4.35) and using (4.19) and (4.20). This completes the proof of Theorem 4.1.

5 Uniform A Priori Estimates

The theorem proved in this section gives a uniform a priori estimate for the entropy weak solution defined in (4.2) of the Cauchy problem (1.1) and (3.1). As we shall see, this estimate justifies some assumptions made in Section 3 and should be useful for obtaining the existence of global weak solutions for the Cauchy problem.

Theorem 5.1. *If $(\rho,\mathbf{m}) \in L^\infty([0,T];L^1(\mathbb{R}^3))$ satisfies the first equation in (1.1) in the sense of distributions, then*

$$\int_{\mathbb{R}^3} \rho(x,t)dx = \int_{\mathbb{R}^3} \rho(x,0)dx =: M, \qquad 0 < t < T. \qquad (5.1)$$

Let $(\rho,\mathbf{m},\Phi)$ be a weak solution defined in Definition 3.1. Suppose $(\rho,\mathbf{m},\Phi)$ satisfies the entropy condition (4.2), $\rho \in L^\infty([0,T];L^1(\mathbb{R}^3)) \cap L^\infty([0,T];L^r(\mathbb{R}^3))$ for some r satisfying $r > 3/2$ and $r \geq \gamma$, $\mathbf{m} \in L^\infty([0,T];L^s(\mathbb{R}^3))$ $(s > 3)$, $(\eta,\mathbf{q}) \in L^\infty([0,T];L^1(\mathbb{R}^3))$, where η and $\mathbf{q}$ are given in (4.3). Moreover, we assume that $(\rho,\mathbf{m})$ has the following additional regularity:

$$\lim_{h\to 0}\int_0^t \int_{\mathbb{R}^3} |\rho(x,\tau+h)-\rho(x,\tau)|dxd\tau = 0, \quad t \in (0,T), a.e. \qquad (5.2)$$

Then

$$E(t) \leq E(0), \qquad 0 < t < T, \qquad (5.3)$$

and if $\gamma > \frac{4}{3}$, *then*

$$H(t) \leq C_1 H(0) + C_2, \qquad 0 < t < T, \tag{5.4}$$

where C_1 *and* C_2 *are two positive constants only depending on* γ *and* M *(cf. (5.1)), where*

$$H(t) = \int_{\mathbb{R}^3} \{\frac{\rho^\gamma}{\gamma - 1} + \frac{|\mathbf{m}|^2}{2\rho} + \frac{1}{8\pi}|\nabla\Phi|^2\}(x,t)dx, \quad t \in [0,T)$$

.

Remark 11. (5.1) and (5.3) justify some assumptions made in Section 3 on the conservation of total mass and non-increase of energy.

Remark 12. The boundedness of $\int_{\mathbb{R}^3} \rho^\gamma(x,t)dx$ was proved in [10] for smooth solutions if $\gamma > 4/3$. Here we prove that this is still true for general week solutions satisfying the entropy condition even without assuming that $\rho \in L^\infty$. In fact, the global existence of radial L^∞-solutions was proved in [34] for (1.1) outside a ball. The blow-up of L^∞-norm of the radial solutions of (1.1) in the entire $\mathbb{R}^3$ space was discussed in [27] and [11], respectively.

Remark 13. Condition (5.2) can be assured by the following condition

$$\lim_{\epsilon \to 0} \sup_{0 \leq \tau \leq T, |y| \leq 1} \int_{\mathbb{R}^3} |\rho(x,\tau) - \rho(x - \epsilon y, \tau)dx = 0, \tag{5.5}$$

if $(\rho, \mathbf{m}) \in L^\infty([0,T]; L^1(\mathbb{R}^3))$; this is proved in the Appendix. Note that (5.2) is the L^1 modulus of continuity in time and (5.5) is the L^1 modulus of continuity in space.

In order to prove this theorem, we begin with the following lemma.

Lemma 5.1. *If* $f \in L^r(\mathbb{R}^3)$ $(r \geq 1)$, *then*

$$Bf \in \begin{cases} L^p(\mathbb{R}^3), \text{ with } 1/p = 1/r - 2/3, & \text{if } r < 3/2, \\ L^\infty(\mathbb{R}^3), & \text{if } r \geq 3/2; \end{cases} \tag{5.6}$$

and

$$\nabla(Bf) \in \begin{cases} L^q(\mathbb{R}^3), \text{ with } 1/q = 1/r - 1/3, & \text{if } r < 3, \\ L^\infty(\mathbb{R}^3), & \text{if } r \geq 3. \end{cases} \tag{5.7}$$

The proof of this lemma follows from the extended Young inequality (cf. [28], p. 32).

Lemma 5.2. *Suppose* $0 \leq \rho \in L^\infty([0,T]; L^1(\mathbb{R}^3))$ *and* $\frac{\mathbf{m}}{\sqrt{\rho}} \in L^\infty([0,T]; L^2(\mathbb{R}^3))$, *then*

$$\mathbf{m} \in L^\infty([0,T]; L^1(\mathbb{R}^3)). \tag{5.8}$$

Proof. Using Hölder inequality, we have

$$\int |\mathbf{m}|dx = \int \sqrt{\rho}\frac{|\mathbf{m}|}{\sqrt{\rho}}dx \leq (\int \rho dx)^{1/2}(\int \frac{|\mathbf{m}|^2}{\rho})^{1/2}. \tag{5.9}$$

Note that (5.9) implies $\mathbf{m} \in L^\infty([0,T]; L^1(\mathbb{R}^3))$. ☐

Remark 14. $\eta \in L^\infty([0,T];L^1(\mathbb{R}^3))$ implies $\frac{\mathbf{m}}{\sqrt{\rho}} \in L^\infty([0,T];L^2(\mathbb{R}^3))$.

Lemma 5.3. *Let $(\rho,\mathbf{m},\Phi)$ be a weak solution defined in Definition 3.1. Suppose $(\rho,\mathbf{m},\Phi)$ satisfies the entropy condition (4.2), $\rho \in L^\infty([0,T];L^1(\mathbb{R}^3)) \cap L^\infty([0,T];L^r(\mathbb{R}^3))$ for some r satisfying $r > 3/2$ and $r \geq \gamma$, $\mathbf{m} \in L^\infty([0,T];L^s(\mathbb{R}^3))$ $(s>3)$, $(\eta,\mathbf{q}) \in L^\infty([0,T];L^1(\mathbb{R}^3))$, where η and $\mathbf{q}$ are given in (4.3). Then, for any $\tau \in [0,T)$, we have*

$$\int_{\mathbb{R}^3} \eta(x,\tau)dx - \int_0^\tau \int_{\mathbb{R}^3} \mathbf{m}\cdot\nabla\Phi dxdt \leq \int_{\mathbb{R}^3} \eta(x,0)dx, \quad \tau \in (0,T), a.e. \tag{5.10}$$

Proof. For a fixed $\tau \in (0,T)$, and small positive ϵ and $R>0$, we define

$$\theta(t) = \begin{cases} 1, & 0 \leq t \leq \tau, \\ -\frac{1}{\epsilon}(t-\tau)+1, & \tau \leq t \leq \tau+\epsilon, \\ 0, & \tau+\epsilon \leq t \leq T, \end{cases} \tag{5.11}$$

and for $x \in \mathbb{R}^3$,

$$\alpha(x) = \begin{cases} 1, & |x| \leq R, \\ -\frac{1}{\epsilon}(|x|-R)+1, & R \leq |x| \leq R+\epsilon, \\ 0, & |x| \geq R+\epsilon. \end{cases} \tag{5.12}$$

Let $\beta(x,t)=\theta(t)\alpha(x)$, then $\beta(x,t)$ is Lipschitz continuous, with compact support in $[0,T) \times \mathbb{R}^3$. Using (4.2), a calculation yields

$$\begin{aligned} &-\frac{1}{\epsilon}\int_\tau^{\tau+\epsilon}\int_{|x|\leq R}\eta(x,t)dxdt - \frac{1}{\epsilon}\int_\tau^{\tau+\epsilon}\int_{R\leq|x|\leq R+\epsilon}\eta(x,t)\alpha(x)dxdt \\ &-\frac{1}{\epsilon}\int_0^{\tau+\epsilon}\int_{R\leq|x|\leq R+\epsilon}(\sum_{j=1}^3 q_j\frac{x_j}{|x|})\theta(t)dxdt \\ &+\int_{|x|\leq R}\eta(x,0)dx + \int_{R\leq|x|\leq R+\epsilon}\eta(x,0)\alpha(x)dx \\ &+\int_0^\tau\int_{|x|\leq R}\mathbf{m}\cdot\nabla\Phi dxdt + \int_\tau^{\tau+\epsilon}\int_{|x|\leq R+\epsilon}\mathbf{m}\cdot\nabla\Phi\beta(x,t)dxdt \geq 0. \end{aligned} \tag{5.13}$$

Since $(\eta,\mathbf{q}) \in L^\infty([0,T],L^1(\mathbb{R}^3)$, we have

$$\lim_{R\to\infty}\int_{R\leq|x|\leq R+\epsilon}\eta(x,t)\alpha(x)dx = 0, \qquad a.e.,\ t\in[0,T], \tag{5.14}$$

$$\lim_{R\to\infty}\int_{R\leq|x|\leq R+\epsilon}\eta(x,0)\alpha(x)dx = 0, \tag{5.15}$$

and

$$\lim_{R\to\infty}\int_{R\leq|x|\leq R+\epsilon}(\sum_{j=1}^3 q_j\frac{x_j}{|x|})\theta(t)dx = 0, \qquad a.e.,\ t\in[0,T]. \tag{5.16}$$

We let $R \to \infty$ in (5.13) to get

$$-\frac{1}{\epsilon}\int_{\tau}^{\tau+\epsilon}\int_{\mathbb{R}^3}\eta(x,t)dxdt + \int_{\mathbb{R}^3}\eta(x,0)dx$$
$$+\int_0^{\tau}\int_{\mathbb{R}^3}\mathbf{m}\cdot\nabla\Phi dxdt + \int_{\tau}^{\tau+\epsilon}\int_{\mathbb{R}^3}\mathbf{m}\cdot\nabla\Phi\beta(x,t)dxdt \geq 0. \tag{5.17}$$

Because $\rho \in L^\infty([0,T]; L^r(\mathbb{R}^3))$ with $r > 3/2$, by (5.7) we have $\nabla\Phi \in L^\infty([0,T]; L^q(\mathbb{R}^3)$ with $q > 3$. It then follows from (5.8), the assumption $\mathbf{m} \in L^\infty([0,T]; L^s(\mathbb{R}^3))$ with $s > 3$ and Holder's inequality that

$$\mathbf{m}\cdot\nabla\Phi \in L^\infty([0,T]; L^1(\mathbb{R}^3)).$$

This implies

$$\lim_{\epsilon\to 0}\int_{\tau}^{\tau+\epsilon}\int_{\mathbb{R}^3}\mathbf{m}\cdot\nabla\Phi\beta(x,t)dxdt = 0.$$

Letting $\epsilon \to 0$ in (5.17), we obtain (5.10). □

Lemma 5.4. *Let* $(\rho, \mathbf{m}, \Phi)$ *be an entropy weak solution defined in Section 4 satisfying the conditions in Lemma 5.3. Then*

$$\partial_t\Phi(x,t) = -\int_{\mathbb{R}^3}\mathbf{m}(y,t)\cdot\nabla_y(\frac{1}{|y-x|})dy. \tag{5.18}$$

Moreover

$$\partial_t\Phi \in L^\infty([0,T]; L^1(\mathbb{R}^3)), \tag{5.19}$$

and

$$\partial_t\Phi \in L^\infty([0,T]\times\mathbb{R}^3). \tag{5.20}$$

Proof. The key is to prove (5.18). Once (5.18) is proved, (5.19) and (5.20) follow from the fact that $\mathbf{m} \in L^\infty([0,T]; L^1(\mathbb{R}^3)) \cap L^\infty([0,T]; L^s(\mathbb{R}^3))$ and the extended Young's inequality (cf. [28], p.32). In order to prove (5.18), we use the fact that $(\rho, \mathbf{m})$ satisfies the first equation of (1.1) in the sense of distributions. For this purpose, we choose a C^∞ function $\delta(z)$ $(z \in \mathbb{R}^1)$ with compact support in the interval $[1,2]$ satisfying $0 \leq \delta(z) \leq 1$ and $\int_{-\infty}^{+\infty}\delta(z)dz = 1$, and let

$$\delta_\epsilon(z) = \frac{1}{\epsilon}\delta(\frac{z}{\epsilon}),\ \alpha_\epsilon(z) = \int_{-\infty}^{z}\delta_\epsilon(s)ds, \quad z \in \mathbb{R}^1, \tag{5.21}$$

for small positive ϵ. For $y \in \mathbb{R}^3$, $0 < \epsilon < \frac{1}{2}$ and $R > 1$, we set

$$f_\epsilon^R(y) = \begin{cases} \alpha_\epsilon(|y|), & |y| \leq \frac{R}{2}+\epsilon, \\ \alpha_\epsilon(R+2\epsilon-|y|), & |y| \geq \frac{R}{2}+\epsilon. \end{cases} \tag{5.22}$$

Then

$$\begin{cases} f_\epsilon^R(y) = 0, & as\ |y| \leq \epsilon,\ or\ |y| \geq R+\epsilon, \\ 0 \leq f_\epsilon^R(y) \leq 1, & as\ \epsilon \leq |y| \leq 2\epsilon,\ or\ R \leq |y| \leq R+\epsilon, \\ f_\epsilon^R(y) = 1, & as\ 2\epsilon \leq |y| \leq R. \end{cases} \tag{5.23}$$

For $x \in \mathbb{R}^3$, we choose

$$g_\epsilon^R(y) = f_\epsilon^R(y-x)\frac{1}{|y-x|}. \tag{5.24}$$

Then $g_\epsilon^R(y) \in C_0^\infty(\mathbb{R}^3)$ for any fixed $x \in \mathbb{R}^3$. Since $(\rho, \mathbf{m})$ satisfies the first equation of (1.1) in the sense of distributions, it is easy to show (see [15] for instance), $\int_{\mathbb{R}^3} \rho(y,t) g_\epsilon^R(y)dy$ is differentiable in t for $t \in [0,T]$ a. e., and satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^3} \rho(y,t) g_\epsilon^R(y)dy = \int_{\mathbb{R}^3} \mathbf{m}(y,t)\cdot\nabla_y g_\epsilon^R(y)dy, \qquad t \in [0,T],\ a.\ e. \tag{5.25}$$

We also let

$$g_\epsilon(y) = \lim_{R\to\infty} g_\epsilon^R(y), \qquad y \in \mathbb{R}^3. \tag{5.26}$$

Then we show (5.18) in the following steps.
Step 1. We show that $\int_{\mathbb{R}^3} \rho(y,t) g_\epsilon(y)dy$ is differentiable for $t \in (0,T]$, a.e., and

$$\frac{d}{dt}\int_{\mathbb{R}^3} \rho(y,t) g_\epsilon(y)dy = \int_{\mathbb{R}^3} \mathbf{m}\cdot\nabla_y g_\epsilon(y)dy, \tag{5.27}$$

for $t \in (0,T], a.e.$
For this purpose, we prove that

$$\frac{1}{h}\int_{\mathbb{R}^3} \frac{\rho(y,t+h)-\rho(y,t)}{h} g_\epsilon^R(y)dy \to \int_{\mathbb{R}^3} \mathbf{m}(y,t)\cdot\nabla_y g_\epsilon^R(y)dy$$
as $h \to 0$ uniformly in R for $R \geq 1$. (5.28)

This is proved as follows. Since $(\rho, \mathbf{m})$ satisfies the first equation of (1.1) in the sense of distributions and $g_\epsilon^R(y) \in C_c^\infty(\mathbb{R}^3)$, it is easy to verify (see [15] for instance),

$$\int_{\mathbb{R}^3} (\rho(y,t+h)-\rho(y,t)) g_\epsilon^R(y)dy = \int_t^{t+h}\int_{\mathbb{R}^3} \mathbf{m}(y,s)\cdot\nabla_y g_\epsilon^R(y)dyds, \tag{5.29}$$

for $[t,t+h] \subset [0,T]$. Thus,

$$\lim_{h\to 0}\int_{\mathbb{R}^3} \frac{(\rho(y,t+h)-\rho(y,t))}{h} g_\epsilon^R(y)dy = \lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^3} \mathbf{m}(y,s)\cdot\nabla_y g_\epsilon^R(y)dyds. \tag{5.30}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^3} \mathbf{m}(y,s)\cdot\nabla_y g_\epsilon^R(y)dyds - \int_{\mathbb{R}^3} \mathbf{m}(y,t)\cdot\nabla_y g_\epsilon^R(y)dy \\
&= \frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^3} (\mathbf{m}(y,s)-\mathbf{m}(y,t))\cdot\nabla_y g_\epsilon^R(y)dyds \\
&= \frac{1}{h}\int_t^{t+h}\int_{\epsilon\leq|y-x|\leq R} (\mathbf{m}(y,s)-\mathbf{m}(y,t))\cdot\nabla_y (\frac{\alpha_\epsilon(|y-x|)}{|y-x|})dyds \\
&+ \frac{1}{h}\int_t^{t+h}\int_{R\leq|y-x|\leq R+\epsilon} (\mathbf{m}(y,s)-\mathbf{m}(y,t))\cdot\nabla_y g_\epsilon^R(y)dyds.
\end{aligned} \tag{5.31}$$

The first term can be handled as follows. For $h > 0$,

$$\begin{aligned}
&|\frac{1}{h}\int_t^{t+h}\int_{\epsilon\le|y-x|\le R}(\mathbf{m}(y,s)-\mathbf{m}(y,t))\cdot\nabla_y(\frac{\alpha_\epsilon(|y-x|)}{|y-x|})dyds| \\
&\le |\frac{1}{h}\int_t^{t+h}\int_{\epsilon\le|y-x|\le R}|\mathbf{m}(y,s)-\mathbf{m}(y,t)|(\frac{\delta_\epsilon(|y-x|)}{|y-x|}+\frac{\alpha_\epsilon(|y-x|)}{|y-x|^2})dyds \\
&\le \frac{2}{\epsilon h}\int_t^{t+h}\int_{\epsilon\le|y-x|\le 2\epsilon}|\mathbf{m}(y,s)-\mathbf{m}(y,t)|\frac{1}{|y-x|}dyds \\
&+\frac{1}{h}\int_t^{t+h}\int_{2\epsilon\le|y-x|\le R}|\mathbf{m}(y,s)-\mathbf{m}(y,t)|\frac{1}{|y-x|^2}dyds \\
&\le \frac{2}{\epsilon^2 h}\int_t^{t+h}\int_{\epsilon\le|y-x|\le 2\epsilon}|\mathbf{m}(y,s)-\mathbf{m}(y,t)|dyds \\
&+\frac{1}{4\epsilon^2 h}\int_t^{t+h}\int_{2\epsilon\le|y-x|\le R}|\mathbf{m}(y,s)-\mathbf{m}(y,t)|dyds. \qquad (5.32)
\end{aligned}$$

The last term in (5.31) can be estimated as follows.

$$\begin{aligned}
&|\frac{1}{h}\int_t^{t+h}\int_{R\le|y-x|\le R+\epsilon}(\mathbf{m}(y,s)-\mathbf{m}(y,t))\cdot\nabla_y g_\epsilon^R(y)dyds| \\
&\le (\frac{1}{\epsilon}+\frac{1}{R})\frac{1}{h}\int_t^{t+h}\int_{R\le|y-x|\le R+\epsilon}|\mathbf{m}(y,s)-\mathbf{m}(y,t)|\frac{1}{|y-x|}dyds \\
&\le (\frac{1}{\epsilon}+\frac{1}{R})\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^3}|\mathbf{m}(y,s)-\mathbf{m}(y,t)|dyds. \qquad (5.33)
\end{aligned}$$

Since we choose $R > 1$, (5.31), (5.32) and (5.33) yield,

$$\begin{aligned}
&|\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^3}\mathbf{m}(y,s)\cdot\nabla_y g_\epsilon^R(y)dyds-\int_{\mathbb{R}^3}\mathbf{m}(y,t)\cdot\nabla_y g_\epsilon^R(y)dy| \\
&\le (\frac{9}{4\epsilon^2}+\frac{1}{\epsilon}+1)\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^3}|\mathbf{m}(y,s)-\mathbf{m}(y,t)|dyds. \qquad (5.34)
\end{aligned}$$

Since $\mathbf{m}\in L^\infty([0,T];L^1(\mathbb{R}^3))$, we have

$$\lim_{h\to 0+}\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^3}|\mathbf{m}(y,s)-\mathbf{m}(y,t)|dyds=0, \quad t\in[0,T],\ a.e. \qquad (5.35)$$

Therefore $\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^3}(\mathbf{m}(y,s)-\mathbf{m}(y,t))\cdot\nabla_y g_\epsilon^R(y)dyds$ converges to zero as $h\to 0+$ uniformly in R for $R > 1$. By a similar approach, we can show that $\frac{1}{h}\int_{t-h}^{t}\int_{\mathbb{R}^3}(\mathbf{m}(y,s)-\mathbf{m}(y,t))\cdot\nabla_y g_\epsilon^R(y)dyds$ converges to zero as $h\to 0-$ uniformly in R for $R > 1$. This verifies (5.28).

(5.27) follows by the following argument, using (5.22) and (5.24).

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\rho(y,t)g_\epsilon(y)dy \\
&= \lim_{h\to 0}\lim_{R\to\infty}\int_{\mathbb{R}^3}\frac{\rho(y,t+h)-\rho(y,t)}{h}g_\epsilon^R(y)dy \\
&= \lim_{R\to\infty}\lim_{h\to 0}\int_{\mathbb{R}^3}\frac{\rho(y,t+h)-\rho(y,t)}{h}g_\epsilon^R(y)dy \\
&= \lim_{R\to\infty}\frac{d}{dt}\int_{\mathbb{R}^3}\rho(y,t)g_\epsilon^R(y)dy \\
&= \lim_{R\to\infty}\int_{\mathbb{R}^3}\mathbf{m}\cdot\nabla_y g_\epsilon^R(y)dy \\
&= \int_{\mathbb{R}^3}\mathbf{m}\cdot\nabla_y g_\epsilon(y)dy. \qquad (5.36)
\end{aligned}
$$

Step 2. In this step, we show that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\rho(y,t)g_\epsilon(y)dy \to \int_{\mathbb{R}^3}\frac{\rho(y,t)}{|y-x|}dy \text{ as } \epsilon\to 0 \\
&\text{uniformly in } t \text{ for } t\in(0,T), \qquad (5.37)
\end{aligned}
$$

and

$$
\lim_{\epsilon\to 0}\int_{\mathbb{R}^3}\mathbf{m}(y,t)\cdot\nabla_y g_\epsilon(y)dy = \int_{\mathbb{R}^3}\mathbf{m}(y,t)\cdot\nabla_y(\frac{1}{|y-x|})dy, \qquad (5.38)
$$

for $t\in(0,T)$.

We prove (5.37) as follows. Since $\rho\in L^\infty([0,T];L^r(\mathbb{R}^3))$ with $r>3/2$ and $r\geq\gamma$, we have, by using Hölder inequality,

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}\frac{\rho(y,t)}{|y-x|}dy - \int_{\mathbb{R}^3}\rho(y,t)g_\epsilon(y)dy| \leq \int_{\epsilon\leq|y-x|\leq 2\epsilon}\frac{\rho(y,t)}{|y-x|}dy \\
&\leq (\int_{\epsilon\leq|y-x|\leq 2\epsilon}\rho^r(y,t)dy)^{1/r}(\int_{\epsilon\leq|y-x|\leq 2\epsilon}\frac{1}{|y-x|^l}dy)^{1/l} \\
&\leq ||\rho||_{L^\infty([0,T];L^r(\mathbb{R}^3))}(\int_\epsilon^{2\epsilon}4\pi s^{2-l}ds)^{1/l}, \qquad (5.39)
\end{aligned}
$$

where $l=\frac{r}{r-1}$. Since $r>3/2$, $l<3$, (5.37) follows. Next (5.38) can be shown as follows.

Since $\mathbf{m} \in L^\infty([0,T];L^s(\mathbb{R}^3))$ for $s>3$, we have

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}\mathbf{m}(y,t)\cdot\nabla_y g_\epsilon(y)dy - \int_{\mathbb{R}^3}\mathbf{m}(y,t)\cdot\nabla_y\frac{1}{|y-x|}dy| \\
&= |\int_{\epsilon\le|y-x|\le 2\epsilon}\mathbf{m}(y,t)\cdot\nabla_y(g_\epsilon(y)-\frac{1}{|y-x|})dy| \\
&= |\int_{\epsilon\le|y-x|\le 2\epsilon}\mathbf{m}(y,t)\cdot\nabla_y(\frac{1}{|y-x|}(\alpha_\epsilon(|y-x|)-1))dy| \\
&\le \int_{\epsilon\le|y-x|\le 2\epsilon}|\mathbf{m}(y,t)|(\frac{1}{|y-x|^2}+\frac{1}{|y-x|}\delta_\epsilon(|y-x|)))dy \\
&\le \frac{2}{\epsilon}\int_{\epsilon\le|y-x|\le 2\epsilon}|\mathbf{m}(y,t)|\frac{1}{|y-x|}dy \\
&\le \frac{2}{\epsilon}||\mathbf{m}||_{L^\infty([0,T];L^s(\mathbb{R}^3))}(\int_{\epsilon\le|y-x|\le 2\epsilon}\frac{1}{|y-x|^q}dy)^{1/q} \\
&\le \frac{2}{\epsilon}||\mathbf{m}||_{L^\infty([0,T];L^s(\mathbb{R}^3))}\left(\int_\epsilon^{2\epsilon}4\pi\tau^{2-s'}d\tau\right)^{1/s'},
\end{aligned}
\tag{5.40}
$$

where $q=\frac{s}{s-1}$. Since $s>3$, then $q<3/2$. Therefore, (5.38) is proved.
By (5.37) and (5.24), we have that $\int_{\mathbb{R}^3}\frac{\rho(y,t)}{|y-x|}dy$ is differentiable with respect to t for $(t,x)\in(0,T)\times\mathbb{R}^3$, a. e. Moreover, by (5.24), (5.37) and (5.38), we obtain,

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3}\frac{\rho(y,t)}{|y-x|}dy &= \frac{d}{dt}(\lim_{\epsilon\to 0}\int_{\mathbb{R}^3}\rho(y,t)g_\epsilon(y)dy) \\
&= \lim_{\epsilon\to 0}\frac{d}{dt}\int_{\mathbb{R}^3}\rho(y,t)g_\epsilon(y)dy = \lim_{\epsilon\to 0}\int_{\mathbb{R}^3}\mathbf{m}(y,t)\cdot\nabla_y g_\epsilon(y)dy \\
&= \int_{\mathbb{R}^3}\mathbf{m}(y,t)\cdot\nabla_y(\frac{1}{|y-x|})dy.
\end{aligned}
\tag{5.41}
$$

This proves (5.18). (5.19) and (5.20) then follows as we showed at the beginning of the proof of this Lemma. □

Proof of Theorem 5.1
We prove Theorem 5.1 in the following steps.

Step 1 In this step, we prove (5.1). This can be proved by using (5.25) in which $g_\epsilon^R(y)$ is replaced by $f_\epsilon^R(y)$, i.e.,

$$
\frac{d}{dt}\int_{\mathbb{R}^3}\rho(y,t)f_\epsilon^R(y)dy = \int_{\mathbb{R}^3}\mathbf{m}(y,t)\cdot\nabla_y f_\epsilon^R(y)dy, t\in[0,T], a.e., \tag{5.42}
$$

where f_ϵ^R is defined in (5.22). We integrate (5.42) to get

$$
\int_{\mathbb{R}^3}\rho(y,t)f_\epsilon^R(y)dy - \int_{\mathbb{R}^3}\rho(y,0)f_\epsilon^R(y)dy = \int_0^t\int_{\mathbb{R}^3}\mathbf{m}(y,s)\cdot\nabla_y f_\epsilon^R(y)dy, \tag{5.43}
$$

By using a same argument as in the proof of Lemma 5.4, we can prove

$$\lim_{\epsilon\to 0}\lim_{R\to\infty}\int_{\mathbb{R}^3}\rho(y,t)f_\epsilon^R(y)dy=\int_{\mathbb{R}^3}\rho(y,t)dy,$$

$$\lim_{\epsilon\to 0}\lim_{R\to\infty}\int_{\mathbb{R}^3}\rho(y,0)f_\epsilon^R(y)dy=\int_{\mathbb{R}^3}\rho(y,0)dy,$$

and

$$\lim_{\epsilon\to 0}\lim_{R\to\infty}\int_0^t\int_{\mathbb{R}^3}\mathbf{m}(y,s)\cdot\nabla_y f_\epsilon^R(y)dy=0.$$

(5.1) follows from (5.43) by letting $R\to\infty$ and $\epsilon\to 0$.

Step 2 In this step, we show that

$$\int_0^t\int_{\mathbb{R}^3}\rho(x,s)\partial_s\Phi(x,s)dxds=\frac{1}{2}\left(\int_{\mathbb{R}^3}(\rho\Phi)(x,t)dx-\int_{\mathbb{R}^3}(\rho\Phi)(x,0)dx\right),\ t\in[0,T). \quad (5.44)$$

This is can be proved as follows.

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\rho(x,s)\partial_s\Phi(x,s)dxds\\
&=\lim_{h\to 0}\frac{1}{h}\int_0^t\int_{\mathbb{R}^3}\rho(x,s)\int_{\mathbb{R}^3}\frac{\rho(y,s+h)-\rho(y,s)}{|x-y|}dydxds\\
&=\lim_{h\to 0}\frac{1}{h}\left(\int_h^{t+h}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(x,s-h)\rho(y,s)}{|x-y|}dydxds-\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(x,s)\rho(y,s)}{|x-y|}dydxds\right)\\
&=\lim_{h\to 0}\frac{1}{h}\int_h^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,s-h)-\rho(x,s))\rho(y,s)}{|x-y|}dydxds\\
&+\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(x,s-h)\rho(y,s)}{|x-y|}dydxds-\lim_{h\to 0}\frac{1}{h}\int_0^h\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(x,s-h)\rho(y,s)}{|x-y|}dydxds\\
&=\lim_{h\to 0}\frac{1}{h}\int_h^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,s-h)-\rho(x,s))\rho(y,s)}{|x-y|}dydxds\\
&+\int_{\mathbb{R}^3}(\rho\Phi)(x,t)dx-\int_{\mathbb{R}^3}(\rho\Phi)(x,0)dx. \quad (5.45)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{h}\int_h^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,s-h)-\rho(x,s))\rho(y,s)}{|x-y|}dydxds\\
&=\frac{1}{h}\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,\tau)-\rho(x,\tau+h))\rho(y,\tau)}{|x-y|}dydxd\tau\\
&+\frac{1}{h}\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,\tau)-\rho(x,\tau+h))(\rho(y,\tau+h)-\rho(y,\tau))}{|x-y|}dydxd\tau\\
&-\frac{1}{h}\int_{t-h}^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,\tau)-\rho(x,\tau+h))\rho(y,\tau+h)}{|x-y|}dydxd\tau. \quad (5.46)
\end{aligned}$$

Since

$$\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}^3}\frac{\rho(y,\tau+h)-\rho(y,\tau)}{|x-y|}dy=-\partial_\tau\Phi(x,\tau)$$

,

$$|\frac{1}{h}\int_{\mathbb{R}^3}\frac{\rho(y,\tau+h)-\rho(y,\tau)}{|x-y|}dy| \le |\partial_\tau\Phi(x,\tau)|+1, \tag{5.47}$$

for small $|h|$. Therefore,

$$|\frac{1}{h}\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,\tau)-\rho(x,\tau+h))(\rho(y,\tau+h)-\rho(y,\tau))}{|x-y|}dydxd\tau|$$
$$\le (||\partial_t\Phi||_{L^\infty([0,T]\times\mathbb{R}^3}+1)\int_0^t\int_{\mathbb{R}^3}|(\rho(y,\tau+h)-\rho(x,\tau)|dyd\tau. \tag{5.48}$$

Then (5.2), (5.20) and (5.48) imply

$$\lim_{h\to 0}\frac{1}{h}\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,\tau)-\rho(x,\tau+h))(\rho(y,\tau+h)-\rho(y,\tau))}{|x-y|}dydxd\tau=0. \tag{5.49}$$

Similarly, we have, for small $|h|$,

$$|\frac{1}{h}\int_{t-h}^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,\tau)-\rho(x,\tau+h))\rho(y,\tau+h)}{|x-y|}dydxd\tau|$$
$$\le (||\partial_t\Phi||_{L^\infty([0,T]\times\mathbb{R}^3}+1)\int_{t-h}^t\int_{\mathbb{R}^3}\rho(y,\tau+h)dyd\tau. \tag{5.50}$$

Since $\rho\in L^\infty([0,T];L^1(\mathbb{R}^3))$, (5.50) implies,

$$\lim_{h\to 0}\frac{1}{h}\int_{t-h}^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,\tau)-\rho(x,\tau+h))\rho(y,\tau+h)}{|x-y|}dydxd\tau=0. \tag{5.51}$$

Hence, (5.46), (5.49) and (5.51) yield

$$\lim_{h\to 0}\frac{1}{h}\int_h^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,s-h)-\rho(x,s))\rho(y,s)}{|x-y|}dydxd\tau$$
$$=\lim_{h\to 0}\frac{1}{h}\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\rho(x,\tau)-\rho(x,\tau+h))\rho(y,\tau)}{|x-y|}dydxd\tau. \tag{5.52}$$

This, together with (5.45), implies (5.44).

Step 3 In this step, we prove (5.3).
Since $\rho\in L^\infty([0,T;L^1(\mathbb{R}^3))\cap L^\infty([0,T];L^r(\mathbb{R}^3))$, where $r>3/2$ and $r\ge\gamma$, we have, in view of (5.7) that

$$\nabla\Phi\in L^\infty([0,T];L^{3/2}(\mathbb{R}^3))\cap L^\infty([0,T];L^\lambda(\mathbb{R}^3)), \tag{5.53}$$

if $r<3$, where $\frac{1}{\lambda}=\frac{1}{r}-\frac{1}{3}$. We also know that $\lambda>3$ if $r>3/2$. Similarly, by (5.7), we have

$$\nabla\Phi\in L^\infty([0,T];L^{3/2}(\mathbb{R}^3))\cap L^\infty([0,T]\times\mathbb{R}^3), \tag{5.54}$$

if $r\ge 3$. Furthermore, because $(\rho,\mathbf{m})$ satisfies the first equation of (1.1) in the sense of distributions, then by a density argument as in [15], in view of (5.19), (5.20), (5.53 and

(5.54), we have,

$$\int_{\mathbb{R}^3}(\rho\Phi)(x,t)dx-\int_{\mathbb{R}^3}(\rho\Phi)(x,0)dx$$
$$=\int_0^t\int_{\mathbb{R}^3}\rho(x,s)\partial_s\Phi(x,s)dxds+\int_0^t\int_{\mathbb{R}^3}\mathbf{m}(x,s)\cdot\nabla\Phi(x,s)dxds, \tag{5.55}$$

for $t\in[0,T)$. This, together with (5.10) and (5.44), implies (5.3), due to the fact

$$E(t)=\int_{\mathbb{R}^3}\eta(x,t)dx-\frac{1}{2}\int_{\mathbb{R}^3}(\rho\Phi)(x,t)dx, \tag{5.56}$$

for $t\in[0,T)$.

Step 4 In this step, we proof (5.4).
First, since $\rho\in L^\infty([0,T];L^1(\mathbb{R}^3)\cap L^\infty([0,T];L^r(\mathbb{R}^3)$ with $r>3/2$, it follows from [22]), [29] and [30] that

$$\frac{1}{2}\int_{\mathbb{R}^3}(\rho\Phi)(x,t)dx=-\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\Phi|^2(x,t)dx,\quad t\in[0,T].$$

Using (2.19), we have, for $\gamma>4/3$

$$\frac{1}{8\pi}|\nabla\Phi|^2\}dx=\int\frac{1}{2}\rho B\rho dx\le C\int\rho^{4/3}dx(\int\rho dx)^{2/3}=M^{2/3}\int\rho^{4/3}dx, \tag{5.57}$$

where $A(\rho)$ is given by (2.3). Taking $p=1$, $q=4/3$, $r=\gamma$, and $a=\frac{\frac{3}{4}\gamma-1}{\gamma-1}$ in Young's inequality (2.17), we obtain,

$$||\rho||_{4/3}\le||\rho||_1^a||\rho||_\gamma^{1-a}=M^a||\rho||_\gamma^{1-a}. \tag{5.58}$$

This is

$$\int\rho^{4/3}dx\le M^{\frac{4}{3}a}(\int\rho^\gamma dx)^b, \tag{5.59}$$

where $b=\frac{1}{3(\gamma-1)}$. Since $\gamma>4/3$, we have $0<b<1$. Therefore, (5.57) and (5.59) imply

$$\int\frac{1}{2}\rho B\rho dx\le C(\gamma-1)^bM^{\frac{4}{3}a+\frac{2}{3}}(\int A(\rho)dx)^b. \tag{5.60}$$

Using the inequality (cf.[15] p. 145)

$$\alpha\beta\le\epsilon\alpha^s+\epsilon^{-t/s}\beta^t, \tag{5.61}$$

if $s^{-1}+t^{-1}=1$ $(s,t>1)$ and $\epsilon>0$, since $b<1$, we can bound $C(\gamma-1)^bM^{\frac{4}{3}a+\frac{2}{3}}(\int A(\rho)dx)^b$ by $\frac{1}{2}\int A(\rho)dx+C_2$, where C_2 is a constant depending only on M and γ (we can take $\epsilon=1/2$ and $s=1/b$ and $t=(1-s^{-1})^{-1}$ in (2.26) since $s>1$ due to $0<b<1$). Therefore,

$$\frac{1}{2}|\int_{\mathbb{R}^3}(\rho\Phi)(x,t)dx|=\frac{1}{8\pi}||\nabla\Phi(\cdot,t)||_2^2\le\frac{1}{2}\int_{\mathbb{R}^3}\frac{\rho^\gamma(x,t)}{\gamma-1}dx+C, \tag{5.62}$$

for $t\in[0,T)$, where C is a constant only depending on $M=\int_{\mathbb{R}^3}\int\rho(x,t)dx=\int_{\mathbb{R}^3}\rho(x,0)dx$ (cf. (5.1)) and γ. This, together with (5.3), implies (5.4).

6 Appendix

In this appendix, we prove the following theorem which is Remark 13 in Section 5.

Theorem If $(\rho, \mathbf{m}) \in L^\infty([0,T]; L^1(\mathbb{R}^3))$ satisfies the first equation of (1.1) in the sense of distributions, then

$$\lim_{\epsilon \to 0} \sup_{0 \le t \le T, |y| \le 1} \int_{\mathbb{R}^3} |\rho(x,t) - \rho(x - \epsilon y, t)| dx = 0, \quad t \in (0,T), a.e., \tag{6.1}$$

implies

$$\lim_{h \to 0} \int_0^t \int_{\mathbb{R}^3} |\rho(x, t+h) - \rho(x,t)| dx = 0, \quad t \in (0,T), a.e. \tag{6.2}$$

Proof. For any fixed $t \in (0,T)$ and small h, we let

$$w(x) = \rho(x, t+h) - \rho(x,t).$$

First, we note that if $\psi(x) \in C^1(\mathbb{R}^3))$ with ψ and $\nabla\psi$ being bounded in $\mathbb{R}^3$, then

$$\int_{\mathbb{R}^3} w(x)\psi(x) dx = \int_t^{t+h} \int_{\mathbb{R}^3} \mathbf{m}(x,s) \cdot \nabla\psi(x) dx ds. \tag{6.3}$$

This is because $(\rho, \mathbf{m}) \in L^\infty([0,T]; L^1(\mathbb{R}^3))$ satisfies the first equation of (1.1) in the sense of distributions. The justification of (6.3) is standard, for instance, see [15]. In view of (6.3), we have

$$|\int_{\mathbb{R}^3} w(x)\psi(x) dx| \le h \sup_{x \in \mathbb{R}^3} |\nabla\psi(x)| ||m||_{L^\infty([0,T]; L^1(\mathbb{R}^3))}. \tag{6.4}$$

We choose ψ as

$$\psi(x) = \int_{\mathbb{R}^3} sgn(x - \epsilon y)\delta(y) dy,$$

where sgn is the sign function, $\delta \in C_0^\infty(\mathbb{R}^3)$ is a smooth function satisfying $0 \le \delta(y) \le 1$, $\int_{\mathbb{R}^3} \delta(y) dy = 1$ and $supp\ \delta \subset \{y \in \mathbb{R}^3 : |y| \le 1\}$. Then $|\nabla\psi| \le \frac{C}{\epsilon}$ for some constant C. Moreover,

$$\begin{aligned}
&|\int_{\mathbb{R}^3} w(x)\psi(x) dx - \int_{\mathbb{R}^3} |w(x)| dx| \\
&= |\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (w(x) - w(x - \epsilon y)) sgn(x - \epsilon y)\delta(y) dy dx| \\
&\le \sup_{|y| \le 1} \int_{\mathbb{R}^3} |w(x) - w(x - \epsilon y)| dy \\
&\le \sup_{|y| \le 1} \int_{\mathbb{R}^3} |\rho(x,t) - \rho(x - \epsilon y, t)| dx + \sup_{|y| \le 1} \int_{\mathbb{R}^3} |\rho(x, t+h) - \rho(x - \epsilon y, t+h)| dx.
\end{aligned} \tag{6.5}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^3}|w(x)|dx\\&\leq \sup_{|y|\leq 1}\int_{\mathbb{R}^3}|\rho(x,t)-\rho(x-\epsilon y,t)|dx+\sup_{|y|\leq 1}\int_{\mathbb{R}^3}|\rho(x,t+h)-\rho(x-\epsilon y,t+h)|dx\\&+\frac{Ch}{\epsilon}||m||_{L^\infty([0,T];L^1(\mathbb{R}^3))}.\end{aligned}\tag{6.6}$$

We let $h\to 0$ first in (6.6), (6.2) follows from (6.1) because ϵ is arbitrary. □

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