

Precautionary Wealth Accumulation: A Positive Third Derivative Is Not Enough

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Abstract

We consider the standard model analyzed in the literature on the life-cycle, permanent-income hypothesis. It is commonly conjectured that expected wealth accumulation in the future increases as earnings risk increases as long as the utility function is increasing, concave and has a positive third derivative. We present a counter example to this conjecture which highlights the importance of the convexity of the savings function or, equivalently, the concavity of the consumption function in a theory of precautionary wealth accumulation.

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1 Introduction

The literature on precautionary wealth accumulation is to a large degree motivated by the possibility that a large part of aggregate wealth accumulation may be due to the presence of uninsured, earnings risk. However, despite sharing a common motivation, the empirical and theoretical literatures on precautionary wealth accumulation have focused on different objects. These objects are highlighted in Figure 1.

[Insert Figure 1(a)-(b) Here]

The empirical literature has focused on establishing whether or not household wealth accumulation increases in a measure of earnings risk, other things equal. Figure 1(a) graphs the situation where more earnings risk leads to more expected wealth accumulation in all periods over the life cycle.¹ One reason why the pattern in Figure 1(a) is central in the literature on precautionary wealth accumulation is that it is the key to establishing whether or not earnings risk may be responsible for a large fraction of aggregate capital accumulation.²

The theoretical literature, in contrast, has focused on when the optimal decision rule for savings (i.e. the savings function) increases with increases in risk. This situation is graphed in Figure 1(b) where wealth carried into the next model period is plotted against a measure of current resources often called cash-on-hand. The key result in the theoretical literature is that this occurs within the framework of maximizing an additively separable expected utility function when the utility function is not only increasing and concave but also has a positive third derivative.³

A central question in the precautionary wealth accumulation literature is what properties of the savings function in Figure 1 (b) are sufficient to produce the pattern in Figure 1 (a). This question was answered by Huggett (2001) for the case where earnings shocks are independent. Huggett shows that when the savings function in Figure 1(b) is increasing and convex in cash-on-hand and when the savings function shifts up when earnings risk increases, then the pattern in Figure 1(a) must hold. The

¹Browning and Lusardi (1996) review the empirical literature.

²The idea is that the law of large numbers can be used to treat the expected wealth profile of one agent as the realized average wealth accumulation profile for many similarly situated agents. Since aggregate wealth is a weighted average of the average cross-sectional wealth holdings of agents of different ages, the upward shift of Figure 1(a) implies that aggregate wealth holding increases as earnings risk increases.

³Leland (1968), Sandmo (1970) and others present this result in the context of a two-period model. Miller (1975, 1976), Sibley (1975), Schechtman (1976) and Amihud and Mendelson (1982) extend this result to multi-period models. Kimball (1990) describes the determinants of the magnitude of the shift of this decision rule for small risks. Caballero (1991) and Weil (1993) present parametric examples with analytic solutions that produce the pattern in Figure 1(a) and (b).

convexity of the savings function turns out to be a key issue. Carroll and Kimball (1996) provide conditions that guarantee that the consumption function is concave or, equivalently, that the savings function is convex. These conditions are stronger than the standard conditions (i.e. $u' > 0, u'' < 0$ and $u''' > 0$) that guarantee that the savings function shifts upward with increases in risk. This leads to a natural open question.

The open question is whether conditions on utility functions beyond the concavity of the utility function and a positive third derivative are essential to guarantee that the expected wealth profile shifts upward with increases in risk. This paper answers this question. In particular, two examples are constructed where $u' > 0, u'' < 0$ and $u''' > 0$ but where the pattern in Figure 1(a) does not hold. The examples highlight how a concave savings function can counteract the effect of the upward shift of the savings function highlighted in Figure 1 (b) in any model with more than two periods.

The upshot of this paper is that economists need to revise their thinking on the fundamental theoretical determinants of precautionary wealth accumulation. Specifically, this paper makes two main points. First, a positive third derivative is not enough for the expected wealth accumulation profile to increase with increases in earnings risk. Second, the reason that a positive third derivative is not enough is that the convexity of the savings function is key in a theory of precautionary wealth accumulation and that convexity is governed by properties beyond a positive third derivative.

The remainder of the paper is organized in two sections. Section 2 describes the standard consumption-savings problem considered in the literature on precautionary savings. Section 3 presents two examples which make the two points described above.

2 Consumption-Savings Problem

An agent maximizes an additively separable expected utility function. The only randomness in the decision problem comes from earnings e_j which are assumed to be drawn independently from age-specific distributions $\pi_{j\theta}$ indexed by a parameter θ . This problem is stated below where it is understood that a single, risk-free asset with gross return $(1 + r) > 0$ is available and that after the last period of life savings must be non-negative (i.e. $a_{J+1} \geq 0$).

$$\begin{aligned} & \text{Max } E[\sum_{j=1}^J u_j(c_j)] \\ & \text{s.t. } c_j + a_{j+1} = a_j(1 + r) + e_j \text{ and } a_{J+1} \geq 0 \end{aligned}$$

A solution to this problem can be described in a couple of different ways. One useful way, that is familiar from the theory of dynamic programming, is with optimal decision rules for consumption $c_j(x; \theta)$ and savings $a_j(x; \theta)$. These decision rules map

the agent's state x at age j into consumption and savings decisions, given earnings distributions indexed by θ . The state of an agent in period j is the amount of cash-on-hand $x_j \equiv a_j(1+r) + e_j$. Another way of describing the solution is by means of functions which map the history of realizations of the shock variable into the settings each period of consumption, savings and so forth. The next section will highlight each of these ways of describing solutions to this problem.

As the essence of this paper is to compare the solution to this problem for earnings distributions differing in risk, we briefly present a standard notion of increasing risk. One earnings process θ is riskier than another earnings process θ' provided that $\pi_{j\theta}$ is riskier than $\pi'_{j\theta}$ each period j in the sense of Rothschild and Stiglitz (1970).⁴

3 Results

We present two examples showing that a positive third derivative is not enough to imply that the expected wealth profile increases with increases in earnings risk. Each example has three model periods. More than two model periods are essential as the pattern in Figure 1(b) implies the pattern in Figure 1(a) in the special case of a two-period model.

3.1 Example 1

In Example 1 the period utility function is of the constant relative risk-aversion class but where the coefficient of relative risk-aversion, or alternatively the elasticity of intertemporal substitution, varies across periods. The agent starts with no financial wealth and faces an interest rate of zero.

Example 1

- $u_j(c) = c^{(1-\sigma_j)}/(1-\sigma_j)$, where $j = 1, 2, 3$ and $\sigma_1 = \sigma_2 = 0.5, \sigma_3 = 2.0$.
- $a_1 = 0.0$ and $r = 0.0$
- no earnings risk - $(e_1, e_2, e_3) = (1.5, 1.5, 0.0)$
- earnings risk - $(e_1, e_2, e_3) = (1.5, 0.0, 0.0), (1.5, 3.0, 0.0)$ with equal probability.

⁴Probability measure $\pi_{j\theta}$ is riskier than $\pi'_{j\theta}$ provided that $\int f(e)d\pi_{j\theta} \leq \int f(e)d\pi'_{j\theta}$ for all concave functions f for which the integrals exist. The definition implies that when two distributions can be ordered by risk then the means must be equal. Rothschild and Stiglitz (1970) provide this and a number of equivalent ways of defining the order of increasing risk. Shaked and Shanthikumar (1994) review the work on stochastic orders.

The solution to this problem is given in Table 1 below. Table 1 first lists the solution for the problem with no earnings risk and then lists the solution with earnings risk.⁵ The solution with earnings risk has two rows for periods 2 and 3. This reflects the fact that all variables in period 2 and beyond are contingent on the earnings realizations in period 2.

Table 1: Solution to Example 1

Period	Earnings	Wealth	Cash-on-Hand	Consumption
j	e_j	a_j	$a_j(1+r) + e_j$	c_j
1	1.5	0.0	1.5	1.0
2	1.5	0.5	2.0	1.0
3	0.0	1.0	1.0	1.0
Period	Earnings	Wealth	Cash-on-Hand	Consumption
j	e_j	a_j	$a_j(1+r) + e_j$	c_j
1	1.5	0.0	1.5	0.5638
2	0.0	0.9362	0.9362	0.2379
	3.0	0.9362	3.9362	2.6592
3	0.0	0.6984	0.6984	0.6984
	0.0	1.2770	1.2770	1.2770

The expected wealth profile corresponding to each earnings process is displayed in Figure 2. It shows that wealth starts out at the same level at age 1. Wealth holding is greater at age 2 with earnings risk than without as the optimal decision rule for savings shifts upwards as earnings risk increases. However, at age 3 the expected wealth holding profiles cross: expected wealth is 0.9877 with earnings risk, compared to 1.0 without earnings risk.

[Insert Figure 2 Here]

⁵The solution is computed by solving the nonlinear equations corresponding to the Euler equations. The solution with earnings risk is necessarily approximate since only four decimals are provided. Plugging in the consumption values for the relevant periods, one can calculate Euler equation residuals (i.e. $u'_j(c_j) - E[u'_{j+1}(c_{j+1})(1+r)]$). The first period's Euler equation has a residual of 6.4×10^{-5} . The two second period Euler equations have residuals of 5.3×10^{-5} and 8.9×10^{-6} respectively. Smaller residuals can easily be obtained with more decimals.

3.1.1 Understanding Example 1

To understand why the profiles in Figure 2 cross it is helpful to consider Figure 3 which graphs the savings function at age 2.⁶ The concavity of the savings function works to depress expected wealth holding at age 3 when the state at age 2 is dispersed. Essentially what is happening in Example 1 is that the Jensen's Inequality effect in Figure 3 is sufficiently strong so as to offset the fact that the mean value of the state at age 2 is higher with earnings risk than without.

[Insert Figure 3 Here]

The remaining unresolved issue is to understand what determines the local concavity or local convexity of the savings function. This issue was explored by Carroll and Kimball (1996) where the focus was on the concavity of the consumption function. Their results highlight the point that the convexity of the savings function, equivalently the concavity of the consumption function, relies on assumptions that are much stronger than a positive third derivative. To understand one of the key points of their analysis, consider interior solutions to the problem of maximizing the function $u(x - a') + V(a')$. By differentiating the Euler equation $u'(x - a(x)) = V'(a(x))$, one finds that $a'(x) = u''/(u'' + V'')$. By differentiating this last result one finds after some algebra that locally the concavity or convexity of the savings function depends on the comparative local curvature properties of the functions u and V . This result is highlighted below where the functions u and V are evaluated at $x - a(x)$ and $a(x)$ respectively.

$$a''(x) \geq (\leq) 0 \iff u'''u'/u''^2 \leq (\geq) V'''V'/V''^2$$

Armed with this insight, it is clear how to construct examples where the savings function is locally concave. One simply focuses on situations where $u'''u'/(u'')^2 \geq V'''V'/(V'')^2$. In Example 1 this was accomplished by changing the coefficient of relative risk aversion across model periods. In particular, between the second and third model periods a problem of the general type above occurs where $u(x - a') = u_2(x - a')$ and $V(a') = u_3(a'(1 + r) + e_3)$. One can calculate that $u'''u'/u''^2 = 1 + 1/\sigma_2$ and $V'''V'/V''^2 = 1 + 1/\sigma_3$. Thus, a situation where $\sigma_2 < \sigma_3$ produces a savings function that is everywhere concave as in Figure 3.

⁶We calculate the savings function $a_2(x; \theta)$ by solving the Euler equation for different values of cash-on-hand x : $u'_2(x - a_2(x; \theta)) = u'_3(a_2(x; \theta)(1 + r) + e_3)(1 + r)$

3.2 Example 2

Example 1 highlights how extra earnings risk may not be converted into more wealth accumulation even when there is precautionary savings in the sense that the decision rule for savings shifts up with earnings risk. Example 2 highlights this same point but, unlike Example 1, does so without assuming that the period utility function changes across model periods. The method relies on considering a period utility function where the curvature, as measured by $u'''u'/(u'')^2$, changes as the consumption level changes. As both the class of constant relative risk aversion and the class of constant absolute risk aversion utility functions have constant values of this curvature measure, it is clear that one must depart from these classes to find a counter example.⁷

Example 2 is based on choosing the period utility function to be the sum of two functions which each are of the constant absolute risk-aversion class. Clearly, the period utility function is increasing, concave and has a positive third derivative as each of its component functions have these properties. In example 2 the interest rate serves the role of moving consumption into different curvature zones across model periods. Alternatively, multiplicative taste shifters (i.e. $u_j(c) = \beta_j u(c)$ for $\beta_j > 0$) could be used to achieve this effect for arbitrary values of the interest rate.

Example 2

- $u_j(c) = -e^{-ac}/a - e^{-bc}/b$, where $j = 1, 2, 3$ and $a = 4.0, b = 0.2$.
- $a_1 = 0.0$ and $r = 1.0$
- no earnings risk - $(e_1, e_2, e_3) = (1.5, 1.5, 0.0)$
- earnings risk - $(e_1, e_2, e_3) = (1.5, 0.0, 0.0), (1.5, 3.0, 0.0)$ with equal probability.

The solution to this problem is given in Table 2. As before, the solution is given first for the problem with no earnings risk, and then for the problem with earnings risk.⁸ Wealth holding starts out at the same level at age 1. Wealth holding is greater at age 2 with earnings risk than without as the savings function shifts upwards as earnings risk increases. However, at age 3 expected wealth is 2.5021 with earnings risk compared to 2.5320 without earnings risk.

⁷For deterministic problems this is clear from the discussion in the last section. For problems with earnings risk, Carroll and Kimball (1996) show that the consumption function will be concave for these classes of utility functions. Thus, the savings function will be convex.

⁸The solution with four decimals is necessarily approximate for both the case with and without earnings risk. For the case with no earnings risk, the first and second period Euler equation residuals are 9.1×10^{-5} and -6.5×10^{-7} , respectively. For the case of earnings risk, the residuals are 7.2×10^{-5} in the first period, and 1.0×10^{-5} and -1.6×10^{-6} respectively in the second period.

Table 2: Solution to Example 2

Period	Earnings	Wealth	Cash-on-Hand	Consumption
j	e_j	a_j	$x_j = a_j(1+r) + e_j$	c_j
1	1.5	0.0	1.5	0.1793
2	1.5	1.3207	4.1413	1.6093
3	0.0	2.5320	5.0640	5.0640
Period	Earnings	Wealth	Cash-on-Hand	Consumption
j	e_j	a_j	$x_j = a_j(1+r) + e_j$	c_j
1	1.5	0	1.5	0.1556
2	0.0	1.3444	2.6887	0.7359
	3.0	1.3444	5.6887	2.6373
3	0.0	1.9528	3.9055	3.9055
	0.0	3.0514	6.1028	6.1028

The crossing of the expected wealth profiles is again due to the concavity of the savings function in period 2. Building on the previous discussion, recall that the savings function will be locally concave at a point x provided that the curvature of the period 2 utility function is greater than the curvature of the period 3 utility function evaluated at the respective consumption levels across periods. Figure 4 plots the curvature as measured by $u'''u'/(u'')^2$ as a function of consumption. The consumption levels in periods 2 and 3 when the agent receives no earnings in period 2 are plotted in Figure 4. Since the curvature is greater at the period 2 consumption level than at the period 3 consumption level, the savings function is locally concave. As in example 1, the concavity of the savings function again counteracts the fact that the mean value of the state in period 2 is higher with earnings risk than without.

[Insert Figure 4 Here]

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