

Precautionary Wealth Accumulation

Mark Huggett*

September 17, 2003

Abstract

When does an individual's expected wealth accumulation profile increase as earnings risk increases? This paper answers this question for multi-period models where earnings shocks are independent over time. Sufficient conditions are stated in terms of properties of a decision rule for savings and, alternatively, in terms of properties of preferences.

JEL Classification: D80, D90, E21

Keywords: Precautionary Wealth, Earnings Risk, Multi-period Models

*I thank Miles Kimball, Edouard Vidon and seminar participants at Cornell, CMSG, Georgetown, George Washington, Maryland, NBER Summer Institute, Penn State, Stockholm School of Economics, Virginia and Yale for helpful comments.

Affiliation: Georgetown University

Address: Economics Department; Georgetown University; Washington DC 20057- 1036

E-mail: mh5@georgetown.edu

Homepage: <http://www.georgetown.edu/faculty/mh5>

Phone: (202) 687- 6683

Fax: (202) 687- 6102

1 Introduction

This paper provides theoretical foundations for when individual and aggregate wealth accumulation increase as earnings risk increases. Stated in terms of a simple picture, the problem addressed in this paper is to find sufficient conditions such that the pattern in Figure 1 holds. Figure 1 describes a situation in which an individual's expected wealth accumulation profile is weakly greater at each age over the life cycle whenever the individual faces one earnings process that is riskier than another earnings process, other things equal.

[Insert Figure 1 Here]

Two important open questions can be answered by finding sufficient conditions for the pattern in Figure 1 to occur. First, there is the question of when theory predicts that households that face more earnings risk have more expected wealth accumulation at any point in the life cycle than households that face less earnings risk. The development of such a theory would be helpful as there is a sizable empirical literature related to this question which is surveyed by Browning and Lusardi (1996). Second, there is the question of when theory predicts that aggregate wealth accumulation increases as earnings risk increases. Much of the interest in the precautionary savings literature is based on the possibility that an important part of aggregate wealth is due to uninsured earnings risk. This question can also be answered with Figure 1. The idea is to treat the expected wealth profile of one individual as the realized average wealth profile of a large number of similarly situated individuals. The law of large numbers justifies this when shocks are independent across individuals and, thus, earnings risk is idiosyncratic. Since aggregate wealth holding is a weighted sum of the average wealth holdings of individuals at different ages, an upward shift of the expected wealth profile of all agents then implies that aggregate wealth holding increases with increases in earnings risk, other things equal.

This paper provides the first general answer for when the pattern in Figure 1 occurs in the context of a multi-period model with independent earnings shocks. The answer is given at two levels: in terms of properties of the decision rule for savings (i.e. the savings function) and in terms of preferences. A savings function maps an individual's current state (e.g. current wealth and earnings) and age into savings or wealth carried into the next model period, given a specific earnings process. Since economists are familiar with the merits of sufficient conditions stated in terms of preferences, I will try to motivate why sufficient conditions stated in terms of the savings function are interesting. First, they are widely applicable. They apply to theories where savings decisions maximize a utility function or are given by a rule-of-thumb. They also apply

to models where there is wage risk and there are both consumption-saving and labor-leisure decisions. Second, insofar as the savings function can be estimated, it is useful for an empirical assessment of the theory of precautionary wealth accumulation. Third, they are useful in the development of preference-based theories of wealth accumulation. The reason is simply that one knows, in advance, what properties of optimal decision rules generate the desired result. Viewed in this way, the focus on decision rules can be seen as a natural first step in an attempt to reverse engineer the pattern in Figure 1.

To understand when the pattern in Figure 1 occurs, it is useful to first have an abstract statement of the main proposition to be established. For this purpose, let x denote an individual's state variable at a point in time. The state represents cash-on-hand and is related to savings or wealth s , earnings z and the interest rate r as follows $x = s(1 + r) + z$. Let λ_j and λ'_j denote the period j distributions (i.e. probability measures) of the state variable from the point of view of the first period when the earnings process is indexed by the parameter θ and θ' respectively. These distributions are induced by the savings functions $s_j(x; \theta)$ and $s_j(x; \theta')$ and corresponding earnings distributions. The main proposition then states that when one earnings process θ is riskier than another θ' (i.e. $\theta \geq_{\Theta} \theta'$) and when the initial distribution λ_1 is ordered above λ'_1 (i.e. $\lambda_1 \succeq \lambda'_1$) then in all future periods j the distribution λ_j is ordered above λ'_j . This proposition is written below. The requirement that the initial distributions are equal (i.e. $\lambda_1 = \lambda'_1$) is relaxed to increase the applicability of the theory.

$$\theta \geq_{\Theta} \theta' \text{ and } \lambda_1 \succeq \lambda'_1 \implies \lambda_j \succeq \lambda'_j \text{ for } j = 1, 2, \dots$$

What kind of order \succeq on distributions might prove useful? Clearly, it must have the property that $\lambda_j \succeq \lambda'_j$ implies that expected wealth holding under λ_j must be at least as great as under λ'_j . In this paper the partial order \succeq employed is called the increasing-convex order. It is defined so that $\lambda \succeq \lambda'$ if and only if $\int_X f d\lambda \geq \int_X f d\lambda'$ for all functions $f(x)$ which are increasing and convex in x . A discussion of why the increasing-convex order is a natural choice will be postponed until the point where this discussion can be better appreciated. The key point to see now is that $\lambda_j \succeq \lambda'_j$ does imply more expected wealth is held in period j with earnings process θ than with θ' . The reason is that cash-on-hand $f(x) = x$ is an increasing and convex function of the state and that expected wealth holding is a monotone increasing function of expected cash-on-hand since $x = s(1 + r) + z$.

The paper presents two main results. The first result says that if the savings function $s_j(x; \theta)$ is increasing in risk θ and is increasing and convex in the state x , then the above proposition holds. This result is tight in that dropping any of these three assumptions allows for the construction of examples where expected wealth accumulation in some period decreases as earnings risk increases. Two fundamental and distinct

properties are key to prove this result. One of these is produced when the savings function increases with risk, whereas the other is produced when the savings function is increasing and convex in the state. Intuitively, increases in mean wealth and wealth dispersion caused by increases in earnings risk and the increase in the savings function are reinforced when the savings function is increasing and convex. In contrast, a decreasing or concave savings function counteracts the effect of the savings function increasing with earnings risk.

The second result provides assumptions on preferences in the expected utility class such that the optimal savings function has the properties assumed in the first result. As a consequence, the expected wealth profile increases as earnings risk increases. It is shown that constant relative risk aversion and constant absolute risk aversion preferences imply this result. It is often conjectured that the expected wealth profile increases with earnings risk when an agent maximizes an additive expected utility function, where the period utility function is increasing and concave in consumption and has a positive third derivative. This paper shows by way of an example that this conjecture is wrong and that stronger conditions on preferences are needed. A concave utility function with a positive third derivative is sufficient to guarantee that the savings function increases in the state and increases with earnings risk but is not sufficient to guarantee that the savings function is convex in the state.

The remainder of the paper is organized as follows. Section 2 reviews the relevant literature. Section 3 describes the framework. Section 4 states and proves the main results. Section 5 discusses the results.

2 Literature Review

This section briefly reviews the theoretical literature which focuses on wealth accumulation as earnings risk increases.¹ The early literature is associated with Leland (1968), Sandmo (1970), Mirman (1971), Rothschild and Stiglitz (1971), Dreze and Modigliani (1972) and Diamond and Stiglitz (1974). They consider a two-period model where future earnings z_2 are random and are drawn from a distribution indexed by the parameter θ . Agents maximize an additive expected utility function $E[u_1(x_1 - s_1) + u_2(s_1(1 + r) + z_2)]$ by choosing the amount of a risk-free asset s_1 with return r to carry to the next period, given initial resources x_1 . The key result is that the optimal decision rule for savings $s_1(x_1; \theta)$ increases as earnings risk increases from

¹Some papers which compare aggregate wealth with earnings risk to aggregate wealth without earnings risk are not reviewed here but are reviewed in Huggett and Ospina (2001).

any initial state x_1 , provided that u'_2 is convex.²

The result follows from the Euler equation below, which ignores corner solutions. The intuition is that, with convex marginal utility, increases in earnings risk increase expected future marginal utility of consumption for any fixed level of savings s_1 and, thus, leads to greater savings carried to the next period. The subsequent literature associated with Miller (1974, 1976), Sibley (1975), Schechtman (1976) and Mendelson and Amihud (1982) shows that the same intuition generalizes to apply to models where preferences over consumption are $E[\sum_{j=1}^J u_j(c_j)]$, earnings shocks are independent and there are arbitrarily many model periods J .³ Collectively, these results are the basis for the claim that u' convex, alternatively $u''' > 0$, is sufficient to generate “precautionary savings”.

$$u'_1(x_1 - s_1) = E[u'_2(s_1(1+r) + z_2)](1+r)$$

The subsequent theoretical literature that relates to wealth accumulation as earnings risk increases can be divided into three groups. The first group presents parametric decision problems having closed-form solutions. Caballero (1991) considers preferences in the expected utility class, whereas van der Ploeg (1993) and Weil (1993) consider preferences that are outside the expected utility class. When earnings are a random walk and the period utility function displays constant absolute risk aversion, Caballero (1991) finds that the expected wealth profile over the life cycle is increasing in earnings risk. Deaton (1992, p.180) and Weil (1993, p. 368) have highlighted some limitations of this result. One of these is that negative consumption is not ruled out. From the perspective of this paper, a key limitation is that since guess and verify is the solution method the general mechanism behind this result is unclear.

A second group calculates the fraction of aggregate wealth holding that is due to the presence of idiosyncratic earnings risk. Skinner (1988), Caballero (1991) and Hubbard et al (1994) use a partial equilibrium framework, whereas Aiyagari (1994) and Huggett (1996) use a general equilibrium framework. These papers, with the exception of Caballero, use constant relative risk aversion preferences and solve the relevant decision problems using computational methods. They find for a number of different specifications of Markovian earnings shocks that aggregate wealth accumulation is greater with earnings risk than without. However, since the results are based on simulations, it is not clear if the qualitative result relies on particular parameter values or holds more generally.

The third group consists of the paper by Kimball (1990) which examines the com-

²Increasing risk is defined as in Rothschild and Stiglitz (1970). This definition induces a partial order on the set of parameters Θ indexing earnings distributions as explained in section 3.

³When $J = \infty$, then $u_j(c_j) = \beta^j u(c_j)$ where $\beta \in (0, 1)$.

parative statics of the two-period precautionary saving problem. One interesting finding that relates to the theme of wealth accumulation as earnings risk increases is that, for infinitesimal amounts of earnings risk, one can say how much the consumption function or the savings function shifts as risk increases. More broadly, Kimball (1990) shows how the tools and findings in Pratt (1964) can be applied to consumption and savings problems.

This paper argues that properties of the savings function beyond the increase of this function with increases in earnings risk are essential in a theory of precautionary wealth accumulation. Specifically, the convexity of the savings function is key for producing the pattern highlighted in Figure 1. Some intuition is now offered for why the convexity of the savings function is important in models with more than two periods.⁴ Consider two individuals who start out in period 1 with the same initial cash-on-hand. One individual faces earnings risk whereas the other does not. Consider the case where the savings function is increasing both in risk and in the state but is not convex in the state (e.g. it is locally concave at some point). In period 2 the distribution of cash-on-hand $x_2 = s_1(x_1; \theta)(1 + r) + z_2$ associated with risky earnings will be dispersed about the point associated with no risk but with a higher mean since the savings function $s_1(x_1; \theta)$ is increasing in risk. In period 3 the expected wealth levels can be reversed by a Jensen's inequality effect due to the local concavity of the savings function when this effect is strong enough to offset the fact that mean cash-on-hand is larger with risk than without.

The importance of a convex savings function in the theory of precautionary wealth accumulation motivates one to ask when is the savings function convex?⁵ Carroll and Kimball (1996, 2001) provide sufficient conditions for the concavity of the consumption function.⁶ The concavity of the consumption function $c_j(x; \theta)$ and the convexity of the savings function $s_j(x; \theta)$ are equivalent statements in models where $c_j(x; \theta) + s_j(x; \theta) = x$.

Carroll and Kimball (1996) show that when the period utility function satisfies $u' > 0$, $u'' < 0$, $u''' > 0$ and $u'''u'/(u'')^2 = k > 0$ the consumption function is concave in cash-on-hand. Utility functions which satisfy this restriction are those in the constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA) class among others. This result holds when the borrowing limits that an agent faces correspond to solvency constraints (i.e. the present value of sure future earnings). Their proof does not cover the case of liquidity constraints (i.e. borrowing limits above the level asso-

⁴Section 4.3 provides an example illustrating this intuition.

⁵It also motivates the empirical question of whether the savings function is convex. Parker (2001) provides empirical support for the proposition that the consumption function is concave using US data.

⁶Zeldes (1989) finds, using computational methods, that the consumption function arising from constant relative risk aversion preferences has a concave shape.

ciated with solvency) since their method of proof relies on high order differentiability of the value function and decision rule. High order differentiability does not hold in the presence of liquidity constraints. Using a different line of attack, Carroll and Kimball (2001) prove that the consumption function is concave for the case of liquidity constraints when the period utility function is in the CARA or CRRA class.

It should be stressed that while Carroll and Kimball (1996 p. 982) highlight the importance of a concave consumption function for a number of interesting issues, they did not foresee the critical role of the convexity of the savings function for the theory of precautionary wealth accumulation. More specifically, they did not anticipate the role of a convex savings function for producing the pattern in Figure 1 which is central in a theory of precautionary wealth accumulation.

3 Framework

The analysis focuses on a savings function $s_j(x; \theta)$ and age-dependent earnings distributions $(\pi_{1\theta}, \dots, \pi_{J\theta})$. A savings function $s_j(x; \theta)$ maps a state x , model period j and earnings process θ into a level of savings carried into the next model period. The state variable x is cash-on-hand and equals $x = s(1+r) + z$, where s is savings, z is earnings and $(1+r) > 0$ is the gross real interest rate.⁷ Age-dependent earnings distributions are indexed by $\theta \in \Theta$. Earnings are independent over periods. The number of model periods J can be finite or infinite.

A savings function and an earnings distribution induce a Markov process on the state variable. The distribution of the state variable is given by a probability measure λ defined on the Borel sets \mathcal{X} of the state space $X = [\underline{x}, \infty)$. The distribution follows a law of motion given by the recursive mapping $T_{j\theta}$. The mapping $T_{j\theta}$ is defined by a transition function $P_{j\theta}(x, B)$, which states the probability that the state next period lies in any set $B \in \mathcal{X}$, given that the period j state is x . The transition function in turn is defined by a savings function and the earnings distributions.⁸

$$\lambda_{j+1}(B) = T_{j\theta}\lambda_j(B) \equiv \int_X P_{j\theta}(x, B) d\lambda_j, \forall B \in \mathcal{X}$$

⁷All the results of the paper can be stated using either a one or a two dimensional state variable ($x = s(1+r) + z$ or $x = (s, z)$). A one dimensional state has the advantage of simplicity, whereas the two dimensional state makes statements about expected wealth more transparent.

⁸The range of the savings function needs to be restricted to guarantee that tomorrow's state lies in the state space. Thus, let $s_j : [\underline{x}, \infty) \rightarrow [\underline{s}_{j+1}, \infty)$. The minimum level of savings \underline{s}_{j+1} is set so that $\underline{x} = \underline{s}_{j+1}(1+r) + \underline{z}_{j+1}$, where \underline{x} is given and \underline{z}_{j+1} is defined as the value of sure earnings in period $j+1$ (i.e. $\underline{z}_{j+1} \equiv \inf \{z : \pi_{j+1\theta}([\underline{z}, z]) > 0\}$).

$$P_{j\theta}(x, B) \equiv \pi_{j+1\theta}(\{z' : s_j(x; \theta)(1+r) + z' \in B\})$$

To complete the description of the framework it remains to define the notions of increasing risk \geq_{Θ} and increasing distributions \succeq . This is done in Definition 1. The notion of increasing risk \geq_{Θ} is a multi-period generalization of that in Rothschild and Stiglitz (1970). One earnings process θ is riskier than another θ' (i.e. $\theta \geq_{\Theta} \theta'$) provided that in every period $\pi_{j\theta}$ is riskier than $\pi_{j\theta'}$ in the Rothschild-Stiglitz sense. The Rothschild-Stiglitz definition is based on the concave order of probability measures.⁹ It states that every risk-averse agent weakly prefers distribution $\pi_{j\theta'}$ to $\pi_{j\theta}$. Clearly, whenever two distributions can be ordered and the means exist the means must be equal. The stochastic order \succeq on distributions of the state variable is called the increasing-convex order.

Definition 1: Let $X = [\underline{x}, \infty)$, $Z = [\underline{z}, \infty)$, \mathcal{X} and \mathcal{Z} be the Borel subsets of X and Z , $\Lambda(X, \mathcal{X})$ and $\Lambda(Z, \mathcal{Z})$ be the set of probability measures on (X, \mathcal{X}) and (Z, \mathcal{Z}) . Let Θ be a nonempty set such that for each $\theta \in \Theta$, $\{\pi_{j\theta} : j = 1, \dots, J\}$ is a collection of probability measures in $\Lambda(Z, \mathcal{Z})$.

- (i) For each $\theta, \theta' \in \Theta$, θ is riskier than θ' (denoted $\theta \geq_{\Theta} \theta'$) provided that for all j , $\int f(z)d\pi_{j\theta} \leq \int f(z)d\pi_{j\theta'}$ for all concave f for which the integrals exist.
- (ii) For each $\lambda, \lambda' \in \Lambda(X, \mathcal{X})$, $\lambda \succeq \lambda'$ provided $\int f(x)d\lambda \geq \int f(x)d\lambda'$ for all increasing and convex functions f for which the integrals exist.

4 Results

4.1 General Theory

Theorem 1 underlies all the results in the paper. Theorem 1 rests on the three key assumptions that are listed in assumption A1.

A1: $s_j(x; \theta)$ is increasing and convex in $x, \forall(j, \theta)$ and increasing in $\theta, \forall(x, j)$.

Theorem 1: Assume A1. For all $\theta, \theta' \in \Theta$ and for all $\lambda_1, \lambda'_1 \in \Lambda(X, \mathcal{X})$ the following holds: $\theta \geq_{\Theta} \theta'$ and $\lambda_1 \succeq \lambda'_1 \implies \lambda_j \succeq \lambda'_j, \forall j$

Proof:

⁹Shaked and Shantikumar (1994) review the literature on stochastic orders. They discuss stochastic dominance, the concave order and the increasing-convex order among others.

Stokey and Lucas (1989, Thm. 9.13) show that $T_{j\theta}$ maps $\Lambda(X, \mathcal{X})$ into itself. Thus, the sequences $\{\lambda_j\}$ and $\{\lambda'_j\}$ are well defined. To prove the theorem it is sufficient to show that $\theta \geq_{\Theta} \theta'$ and $\lambda_j \succeq \lambda'_j \Rightarrow T_{j\theta}\lambda_j \succeq T_{j\theta'}\lambda'_j$. This is established in two steps. Step 1 shows that $T_{j\theta}$ preserves order (i.e. $\lambda \succeq \lambda' \Rightarrow T_{j\theta}\lambda \succeq T_{j\theta}\lambda'$). Step 2 shows that $T_{j\theta}$ increases in θ (i.e. $\theta \geq_{\Theta} \theta' \Rightarrow T_{j\theta}\lambda \succeq T_{j\theta'}\lambda, \forall \lambda$). The result then follows from the transitivity of \succeq after combining these results to get $T_{j\theta}\lambda_j \succeq T_{j\theta}\lambda'_j \succeq T_{j\theta'}\lambda'_j$.

Step 1. Show that $\lambda \succeq \lambda' \Rightarrow T_{j\theta}\lambda \succeq T_{j\theta}\lambda'$. The conclusion of this implication is equivalent to the two statements below. The leftmost equivalence follows from the definition of \succeq , whereas the rightmost follows from Stokey and Lucas (1989, Thm. 8.3). In these statements $E_{j\theta}[f|x] \equiv \int f(x')P_{j\theta}(x, dx')$ and the class of functions f are those that are increasing and convex in x and for which the integrals exist.

$$T_{j\theta}\lambda \succeq T_{j\theta}\lambda' \Leftrightarrow \int f dT_{j\theta}\lambda \geq \int f dT_{j\theta}\lambda' \Leftrightarrow \int E_{j\theta}[f|x]d\lambda \geq \int E_{j\theta}[f|x]d\lambda'$$

Since $\lambda \succeq \lambda'$, the rightmost inequality above holds if $E_{j\theta}[f|x] = \int f(s_j(x; \theta)(1+r) + z')\pi_{j+1\theta}(dz')$ is an increasing and convex function of the state x . This holds for two reasons. First, the composition of increasing functions is increasing and the composition of an increasing, convex function f with a convex function $s_j(x; \theta)(1+r) + z'$ preserves convexity (Rockafellar (1970, Thm. 5.1)). Second, integration preserves each of these properties.

Step 2. Show that $\theta \geq_{\Theta} \theta' \Rightarrow T_{j\theta}\lambda \succeq T_{j\theta'}\lambda, \forall \lambda$. The conclusion of this implication is equivalent to the two expressions below for the reasons given in step 1.

$$T_{j\theta}\lambda \succeq T_{j\theta'}\lambda \Leftrightarrow \int f dT_{j\theta}\lambda \geq \int f dT_{j\theta'}\lambda \Leftrightarrow \int E_{j\theta}[f|x]d\lambda \geq \int E_{j\theta'}[f|x]d\lambda$$

The rightmost inequality above holds if $E_{j\theta}[f|x]$ is an increasing function of θ . This in turn follows from the two inequalities below. The topmost inequality holds as $f(s_j(x; \theta)(1+r) + z') \geq f(s_j(x; \theta')(1+r) + z')$ for all z' since $s_j(x; \theta)$ is increasing in θ . The bottommost inequality follows from $\theta \geq_{\Theta} \theta'$, since $f(s_j(x; \theta')(1+r) + z')$ is convex in z' and thus $-f(s_j(x; \theta')(1+r) + z')$ is concave in z' .

$$\begin{aligned} \int f(s_j(x; \theta)(1+r) + z')\pi_{j+1\theta}(dz') &\geq \int f(s_j(x; \theta')(1+r) + z')\pi_{j+1\theta}(dz') \\ &\geq \int f(s_j(x; \theta')(1+r) + z')\pi_{j+1\theta'}(dz') \quad \square \end{aligned}$$

4.2 Application

The literature that formalizes the life-cycle, permanent-income hypothesis assumes that an agent maximizes an additively separable expected utility function $E[\sum_{j=1}^J \beta^{j-1}u(c_j)]$. An agent receives earnings z_j in period j . Earnings are drawn independently from age-specific distributions $\pi_{j\theta}$ indexed by θ . Each period j the agent divides cash-on-hand

x_j between consumption c_j and savings s_j so that $c_j + s_j = x_j$. Savings receive a gross, risk-free return $(1 + r)$ and must lie above period-specific borrowing limits $\underline{s} \equiv (\underline{s}_1, \dots, \underline{s}_{J+1})$. In this section an agent's lifetime J is assumed to be finite.

The dynamic programming formulation of this decision problem is given below. The state variable equals $x = s(1 + r) + z$ and lies in the period-specific state space $X_j \equiv [\underline{x}_j, \infty)$. The expectations operator has a subscript θ to reflect the distribution used in this operation. The value function is set to zero after the last period of life (i.e. $V_{J+1}(x; \theta) = 0$). The functions $c_j(x; \theta)$ and $s_j(x; \theta)$ denote the optimal decision rules for consumption and savings solving this problem.¹⁰

$$V_j(x; \theta) = \max u(x - s') + \beta E_\theta[V_{j+1}(s'(1 + r) + z'; \theta)]$$

subject to $s' \in \Gamma_j(x) \equiv \{s' : \underline{s}_{j+1} \leq s' \leq x\}$

To specify the relation between the minimum values of cash-on-hand and the implied values of the period-specific borrowing limits, it will be useful to define the sure component of earnings in a period (see Miller (1974)). Let $\underline{z}_j \equiv \inf \{z : \pi_{j\theta}([0, z]) > 0\}$ denote sure earnings in period j for earnings process θ . Given a vector $\underline{x} = (\underline{x}_1, \dots, \underline{x}_J)$ of minimum values of cash-on-hand, define the vector $\underline{s} = (\underline{s}_1, \dots, \underline{s}_{J+1})$ so that $\underline{x}_j = \underline{s}_j(1 + r) + \underline{z}_j$ and that no borrowing is allowed in the last period of life (i.e. $\underline{s}_{J+1} = 0$). The vector \underline{x} is restricted so that each period budget sets are nonempty (i.e. $\underline{s}_{j+1} \leq \underline{x}_j$) and in the last period $\underline{x}_J \geq 0$. This implies that any debt can be repaid with sure future earnings (i.e. $\underline{x}_j \geq -\sum_{k>j} \underline{z}_k / (1 + r)^{k-j}$). An important case is where \underline{x}_j equals the negative of the present value of sure future earnings. This case is called the case of a *solvency* constraint. In contrast, the case of a *liquidity* constraint occurs when the settings of \underline{x}_j are above these values.

4.2.1 Expected Wealth Profiles

Theorem 2 provides assumptions on preferences such that the optimal decision rule for savings has all the properties assumed in Theorem 1. Thus, under these assumptions, the expected wealth profile increases with increases in earnings risk. More generally, the expectation of any increasing and convex function of the state is also increasing in earnings risk. The period utility functions considered in Theorem 2 lie in two classes. Constant relative risk aversion (CRRA) utility functions satisfy $u(c) = c^{1-\sigma}/(1-\sigma)$ for $\sigma > 0$ and $\sigma \neq 1$ and $u(c) = \log(c)$ for $\sigma = 1$. Constant absolute risk aversion (CARA) utility functions satisfy $u(c) = -(1/a)e^{-ac}$ for $a > 0$.

¹⁰The value functions and decision rules depend on the vector of minimum values of cash-on-hand \underline{x} . This dependence is suppressed when it is not essential.

B1 $u(c)$ is increasing, continuous and strictly concave and $\beta, (1+r) > 0$.

B2 $\pi_{j\theta}$ are probability measures on the Borel sets of $Z = [0, \bar{z}]$.

B3 Condition (a) or (b) holds:

(a) $u(c)$ is in the CRRA class

(b) $u(c)$ is in the CARA class and $\beta(1+r) \leq 1$.

Lemma 1: Assume B1-3.

(i) There exists a unique optimal decision rule $s_j(x; \theta)$.

(ii) $s_j(x; \theta)$ is increasing in x

(iii) $s_j(x; \theta)$ is increasing in θ .

(iv) $s_j(x; \theta)$ is convex in x .

(v) $s_j(x; \theta, \underline{x})$ is increasing in \underline{x} .

Proof: See the Appendix.

Theorem 2: Assume B1-3 and that the minimum values of cash-on-hand are the same for earnings processes differing in earnings risk. Then the expected wealth profile increases with increases in earnings risk, other things equal.

Proof: Follows from Theorem 1 and Lemma 1 (i)-(iv) once one allows the state space X in Theorem 1 to change with age due to the fact that the minimum values of cash-on-hand are age-dependent. \square

Which assumptions are key for which issues? Lemma 1 (i)-(ii) states that the savings function $s_j(x; \theta)$ exists and increases in the state x . This follows from assumptions B1-2 using standard results from the theory of dynamic programming and the theory of monotone comparative statics. Lemma 1 (iii) states that $s_j(x; \theta)$ increases in θ . The intuition behind this result was presented in section 2. The proof makes use of assumption B1-2 and the assumption that u' is convex and consumption is interior. The remaining property needed to prove Theorem 2 is that the savings function $s_j(x; \theta)$ is convex in x . The proof makes full use of assumption B3.¹¹ The proof of convexity follows Carroll and Kimball (2001). This important result was discussed previously in section 2.

Theorem 3 generalizes the result in Theorem 2. Specifically, Theorem 3 relaxes the requirement that the minimum values of cash-on-hand must be equal across decision

¹¹The requirement in assumption B3 condition (b) that $\beta(1+r) \leq 1$ needs some explanation. CARA utility functions allow a corner solution with zero consumption for sufficiently high interest rates. This uninteresting case is often ruled out by assuming infinite marginal utility at zero consumption; however, this assumption rules out CARA utility. The reason that it is important to rule out a corner solution is that then the consumption function would not be concave and the savings function would not be convex. Assuming $\beta(1+r) \leq 1$ eliminates the possibility of a corner solution.

problems which differ in earnings risk. Theorem 3 allows one to compare expected wealth profiles when earnings risk differs and when an agent can borrow more when faced with less earnings risk. An earnings process having more risk than another either has the same or a strictly lower present value of sure future earnings.

Theorem 3 follows from Theorem 2 together with one additional argument. The additional argument is that for a fixed earnings process when the limits on cash-on-hand are reduced (i.e. loosened) the expected wealth profile at each age over the life cycle is weakly reduced. This follows since the savings function shifts downward as the minimum value of cash-on-hand falls and, hence, as borrowing constraints are loosened. Lemma 1(v) establishes that this result follows from assumptions B1-2 and, in particular, the concavity of the period utility function. In Lemma 1(v) the savings function $s_j(x; \theta, \underline{x})$ explicitly allows for dependence on $\underline{x} = (\underline{x}_1, \dots, \underline{x}_J)$. Previously, this dependence was not highlighted as \underline{x} was fixed as the other variables were varied.

Theorem 3: Assume B1-3 and that riskier earnings processes have tighter limits on cash-on-hand. Then the expected wealth profile increases with increases in earnings risk, other things equal.

Proof: Consider two earnings processes θ and θ' such that $\theta \geq_{\Theta} \theta'$. Theorem 2 shows that the result holds when the limits on cash-on-hand are the same (i.e. $\underline{x}_j = \underline{x}'_j, \forall j$). Lemma 1 (v) in conjunction with Lemma 1 (ii) implies that when the limits corresponding to the earnings process with less earnings risk (i.e. θ') are loosened or reduced then for any realization of the life-cycle earnings path the realized wealth path is weakly less than before. Therefore, the expected wealth profile for earnings process θ' falls as the vector \underline{x}' falls. \square

4.3 An Example

An open question is whether assumptions on preferences beyond a positive third derivative of the period utility function are needed to guarantee that the expected wealth profile increases as earnings risk increases. This section answers this question by providing an example showing that further assumptions are essential. The agent lives three model periods and maximizes expected utility $E[\sum_{j=1}^3 u_j(c_j)]$. The agent starts out with no wealth ($s_1 = 0$) and faces a zero interest rate ($r = 0$). Two earnings processes are considered: one with no risk and the other with risk. The solution is given in Table 1 which first lists the solution with no earnings risk and then lists the solution with earnings risk.¹² The solution with earnings risk has two rows for periods 2 and 3.

¹²The solution is computed by solving the Euler equations. The solution with earnings risk is necessarily approximate since only four decimals are used. Plugging in the consumption values, one can calculate Euler

This reflects the fact that these variables are contingent on the earnings realizations in period 2.

Example

- $u_j(c) = c^{(1-\sigma_j)}/(1-\sigma_j)$, where $\sigma_1 = \sigma_2 = 0.5$ and $\sigma_3 = 2.0$.
- no earnings risk: $(z_1, z_2, z_3) = (1.5, 1.5, 0.0)$
- earnings risk: $(z_1, z_2, z_3) = (1.5, 0.0, 0.0), (1.5, 3.0, 0.0)$ with equal probability.

[Insert Table 1 Here]

Table 1 shows that expected wealth in period 3 is smaller with earnings risk as compared to no earnings risk. Why? Theorems 1-3 and the discussion of Lemma 1 tells one that if $u'_j > 0$, $u''_j < 0$ and $u'''_j > 0$ then the savings function in period 2 needs to be non-convex somewhere for this to hold. Figure 2 graphs this savings function. Figure 2 shows that the Jensen's Inequality effect produced by a concave savings function is sufficiently strong to offset the fact that the mean value of cash-on-hand at age 2 is higher with earnings risk than without.

[Insert Figure 2 Here]

What determines the local concavity or convexity of the savings function? Consider interior solutions to the problem of maximizing $u(x - s') + V(s')$. By differentiating the Euler equation $u'(x - s(x)) = V'(s(x))$, one finds that $s'(x) = u''/(u'' + V'')$. Differentiating this last result produces a necessary and sufficient condition for the local concavity or convexity of the savings function. The condition presented below, compares a property of u and V evaluated at $x - s(x)$ and $s(x)$ respectively. Armed with this insight, it is clear why the savings function in Figure 2 is concave. In particular, in period 2 a problem of the general type described above occurs where $u(x - s') = u_2(x - s')$ and $V(s') = u_3(s'(1 + r) + z_3)$. Since $u'''u'/(u'')^2 = 1 + 1/\sigma_2$ and $V'''V'/(V'')^2 = 1 + 1/\sigma_3$, the savings function is concave when $\sigma_2 < \sigma_3$.

$$s''(x) \geq (\leq) 0 \iff u'''u'/(u'')^2 \leq (\geq) V'''V'/(V'')^2$$

In summary, given preferences $E[\sum_{j=1}^3 u_j(c_j)]$ where $u'_j > 0$, $u''_j < 0$ and $u'''_j > 0$ and given earnings risk only in the second period, to produce the result in Table 1 it is

equation residuals (i.e. $u'_j(c_j) - E[u'_{j+1}(c_{j+1})(1 + r)]$). The residual for $j = 1$ is 6.4×10^{-5} , whereas the residuals for $j = 2$ are 5.3×10^{-5} and 8.9×10^{-6} respectively.

essential that $u_j'''u_j'/(u_j'')^2$ varies either across model periods j or across consumption levels. The example above does this by varying the coefficient of relative risk aversion across periods. When the period utility function does not change across model periods (i.e. when $u_j = u, \forall j$), then one needs $u'''u'/(u'')^2$ to vary across consumption levels. This is impossible with CARA or CRRA preferences. However, the result in Table 1 can occur when the period utility function is a weighted sum of two CARA or two CRRA utility functions as is shown in Huggett and Vidon (2002).

5 Discussion

The discussion is in the form of several Remarks.

Remark 1:

One question which has not been fully addressed up to this point is why the paper chooses \succeq to be the increasing-convex order? One response is simply to note that the increasing-convex order does lead to a useful characterization. A more complete response would discuss why the increasing-convex order seemed to be promising in the first place. One simple bit of intuition is that the precautionary wealth accumulation problem involves comparing distributions differing in both mean and dispersion. Intuitively, if the pattern in Figure 1 is to hold, then distributions associated with more risk have higher mean wealth but may also have higher wealth dispersion. The increasing-convex order is designed to compare such distributions, whereas other familiar stochastic orders such as first order stochastic dominance are not. Another bit of intuition builds on the discussion in section 2 and the example in section 4.3. There it was clear that when the savings function $s_j(x; \theta)$ was either not increasing or not convex in the state x , then expected wealth holding may decrease in some period as risk increases even when $s_j(x; \theta)$ increases in θ . This also suggests a key role for a stochastic order based on increasing and convex functions.

Remark 2:

The proof of Theorem 1 provides insight into the respective roles of the three key assumptions made on decision rules. To see this, recall that the proof of Theorem 1 is based on showing that $\theta \geq_{\Theta} \theta'$ and $\lambda_j \succeq \lambda'_j$ imply $T_{j\theta}\lambda_j \succeq T_{j\theta'}\lambda'_j$. This is accomplished by showing that the following holds: $T_{j\theta}\lambda_j \succeq T_{j\theta}\lambda'_j \succeq T_{j\theta'}\lambda'_j$. The leftmost relationship states that $T_{j\theta}$ is monotone or order preserving in that if in period j one distribution is larger than another then in period $j + 1$ the distributions resulting from applying the map $T_{j\theta}$ will also have this property. The rightmost relationship states that $T_{j\theta}$ is increasing in θ in that any fixed distribution is mapped into a larger distribution as θ increases.

The role of the three key assumptions is clear. The assumption that the decision rule is increasing and convex in the state is used to establish that $T_{j\theta}$ is monotone or order preserving. The example in section 4.3 highlighted how order is not preserved when the decision rule does not satisfy these assumptions. The assumption that the decision rule increases in risk is used to establish that the map $T_{j\theta}$ is increasing in θ . Finally, it should be stressed that these two properties of the map $T_{j\theta}$ are not only sufficient for the proposition contained in Theorem 1 to hold but also necessary if this is to hold starting in any period from any distributions that can be ordered. Huggett (2003) shows that this is true when \geq_{Θ} is any reflexive binary relation on parameters and \succeq is any reflexive and transitive binary relation on distributions. Thus, these two properties of the map $T_{j\theta}$ are fundamental in building a theory of precautionary wealth accumulation or, more generally, in understanding when comparative dynamics are monotone in models with a recursive structure.

Remark 3:

An open question for future work is whether the results of this paper can be extended to situations where earnings follow a Markov process. This is important as the empirical literature on earnings has characterized that the stochastic component of earnings variation has both a purely temporary as well as a persistent component. It has been conjectured that the persistent component is an especially important determinant of precautionary wealth accumulation with the intuition that a given change in the persistent component may be associated with a much larger shift in the distribution of the present value of earnings than the same change in the temporary component. On the basis of Remark 2, it is clear that an analysis of the Markov shock case will have much in common with the independent shock case considered in this paper.

References

- AIYAGARI, R. (1994). “Uninsured Idiosyncratic Risk and Aggregate Savings”, *Quarterly Journal of Economics*, 109, 659- 84.
- BROWNING, M. and A. LUSARDI (1996). “Household Saving: Micro Theories and Micro Facts”, *Journal of Economic Literature*, 34, 1797- 1855.
- CABALLERO, R. (1991). “Earnings Uncertainty and Aggregate Wealth Accumulation”, *American Economic Review*, 81, 859-71.
- CARROLL, C. and M. KIMBALL (1996). “On the Concavity of the Consumption Function”, *Econometrica*, 64, 981-92.
- CARROLL, C. and M. KIMBALL (2001). “Liquidity Constraints and Precautionary Saving”, manuscript.
- DEATON, A. (1992). *Understanding Consumption*, Clarendon Press, Oxford.
- DIAMOND, P. and J. STIGLITZ (1974). “Increases in Risk and in Risk Aversion”, *Journal of Economic Theory*, 8, 337-60.
- DREZE, J. and F. MODIGLIANI (1972). “Consumption Decisions Under Uncertainty”, *Journal of Economic Theory*, 5, 308-35.
- HARDY, G., LITTLEWOOD, J. and G. POLYA (1967). *Inequalities*, Cambridge University Press, second edition.
- HUBBARD, R., SKINNER, J. and S. ZELDES (1994). “The Importance of Precautionary Motives in Explaining Individual and Aggregate Saving”, *Carnegie-Rochester Conference Series on Public Policy*, 40, 59-126.
- HUGGETT, M. (1996). “Wealth Distribution in Life-Cycle Economies”, *Journal of Monetary Economics*, 38, 469-94.
- HUGGETT, M. and S. OSPINA (2001). “Aggregate Precautionary Savings: When is the Third Derivative Irrelevant?”, *Journal of Monetary Economics*, 48, 373- 96.
- HUGGETT, M. and E. VIDON (2002). “Precautionary Wealth Accumulation: A Positive Third Derivative is Not Enough”, *Economics Letters*, 76, 323-329.
- HUGGETT, M. (2003). “When Are Comparative Dynamics Monotone?”, *Review of Economic Dynamics*, 6, 1-11.
- KIMBALL, M. (1990). “Precautionary Saving in the Small and in the Large”, *Econometrica*, 58, 53-73.
- LELAND, H. (1968). “Savings and Uncertainty: The Precautionary Demand for Saving”, *Quarterly Journal of Economics*, 82, 465-73.

- MENDELSON, H. and Y. AMIHUD (1982). "Optimal Consumption Policy Under Uncertain Income", *Management Science*, 28, 683-97.
- MILLER, B. (1974). "Optimal Consumption with a Stochastic Income Stream", *Econometrica*, 42, 253-266.
- MILLER, B. (1976). "The Effect on Optimal Consumption of Increased Uncertainty in Labor Income in the Multiperiod Case", *Journal of Economic Theory*, 13, 154-167.
- MIRMAN, L. (1971). "Uncertainty and Optimal Consumption Decisions", *Econometrica*, 39, 179-185.
- PARKER, J. (2001). "The Consumption Function Re-Estimated", Princeton University, manuscript.
- PRATT, J. (1964). "Risk Aversion in the Small and in the Large", *Econometrica*, 32, 122-36.
- van der PLOEG, F. (1993). "Risk Aversion, Intertemporal Substitution and Consumption: The CARA-LQ Problem", *Review of Economic Studies*, 60, 367-83.
- ROCKAFELLAR, R. (1970). *Convex Analysis*, Princeton University Press, Princeton.
- ROTHSCHILD, M. and J. STIGLITZ (1970). "Increasing Risk I: A Definition", *Journal of Economic Theory*, 2, 225-43.
- ROTHSCHILD, M. and J. STIGLITZ (1971). "Increasing Risk II: Its Economic Consequences", *Journal of Economic Theory*, 3, 66-84.
- SANDMO, A. (1970). "The Effect of Uncertainty on Saving Decisions", *Review of Economic Studies*, 37, 353-60.
- SCHECHTMAN, J. (1976). "An Income Fluctuation Problem", *Journal of Economic Theory*, 12, 218-41.
- SHAKED, M. and J. SHANTHIKUMAR (1994). *Stochastic Orders and Their Applications*, Academic Press, New York.
- SIBLEY, D. (1975). "Permanent and Transitory Income Effects in a Model of Optimal Consumption with Wage Uncertainty", *Journal of Economic Theory*, 11, 68-82.
- SKINNER, J. (1988). "Risky Income, Life Cycle Consumption, and Precautionary Saving", *Journal of Monetary Economics*, 22, 14-35.
- STOKEY, N. and R. LUCAS (1989). *Recursive Methods in Economic Dynamics*, Harvard University Press.

TOPKIS, D. (1998). *Supermodularity and Complementarity*, Princeton University Press.

WEIL, P. (1993). "Precautionary Savings and the Permanent Income Hypothesis", *Review of Economic Studies*, 60, 367-83.

ZELDES, S. (1989). "Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence", *Quarterly Journal of Economics*, 104, 275-98.

A Appendix

The proof of Lemma 1 (iii)-(iv) uses results (1)-(2) below. Proofs are easily adapted from Schechtman (1976, Theorem 1.3 and Corollary 1.4) and make use of the fact that consumption is interior.

- (1) V_j is differentiable in the state and $V_j'(x; \theta) = u'(c_j(x; \theta))$.
- (2) A necessary condition for maximization (i.e. the Euler equation) is $u'(c_j(x; \theta)) \geq \beta(1+r)E_\theta[V_{j+1}'(s_j(x; \theta)(1+r) + z'; \theta)]$ and $=$ if $s_j(x; \theta) > \underline{s}_{j+1}$.

Lemma 1: Assume B1-3.

- (i) *There exists a unique optimal decision rule $s_j(x; \theta)$.*
- (ii) *$s_j(x; \theta)$ is increasing in x*
- (iii) *$s_j(x; \theta)$ is increasing in θ .*
- (iv) *$s_j(x; \theta)$ is convex in x .*
- (v) *$s_j(x; \theta, \underline{x})$ is increasing in \underline{x} .*

Proof of Lemma 1 (i):

Let u satisfy B1. The extension for the case where $u(c) = c^{(1-\sigma)}/(1-\sigma)$ and $\sigma \geq 1$ is not provided but can be established. For $j = J$ the objective is continuous and the constraint set is nonempty and compact for all $x \in X_J$. Thus, there is a solution for each $x \in X_J$. The solution $s_J(x; \theta)$ is unique since the objective is strictly concave and the constraint set $\Gamma_J(x)$ is convex. As $\Gamma_J(x)$ is a continuous correspondence, the Theorem of the Maximum (Stokey and Lucas (1989, Thm. 3.6)) implies that $(V_J(x; \theta), s_J(x; \theta))$ are continuous in x . Backwards recursion and repeating the above argument (using Stokey and Lucas (1989, Lemma 9.5)) gives unique solutions $(V_j(x; \theta), s_j(x; \theta))$ that are continuous in x for $j = 1, 2, \dots, J$, provided that the objective is strictly concave. The strict concavity of the objective in the control follows from the strict concavity of u and the concavity of V_j . The concavity of V_j follows a standard argument (e.g. Stokey and Lucas (1989, Thm. 4.8)). \square

Proof of Lemma 1 (ii) and (v):

To prove that $s_j(x; \theta, \underline{x})$ is increasing jointly in (x, \underline{x}) define $\hat{x} = (x, \underline{x})$ and note that $s_j(x; \theta, \underline{x})$ is the argmax of $f_j(\hat{x}, y)$ over the constraint set $Y_j(\hat{x}) \equiv \{y : \underline{s}_{j+1} \leq y \leq x\} \subseteq Y$, where \underline{s}_{j+1} is defined by $\underline{x}_{j+1} = \underline{s}_{j+1}(1+r) + \underline{z}_{j+1}$. Throughout the proof the dependence of the relevant functions and operations on θ is omitted.

$$f_j(\hat{x}, y) \equiv u(x - y) + \beta E[V_{j+1}(y(1+r) + z'; \underline{x})]$$

Lemma 1 (ii) and (v) then follows from Topkis (1998, Thm. 2.8.2) when (a) \hat{X}_j, Y are lattices, (b) $S_j \equiv \{(\hat{x}, y) : \hat{x} \in \hat{X}_j, y \in Y_j(\hat{x})\}$ is sublattice of $\hat{X}_j \times Y$ and (c) $f_j(\hat{x}, y)$

is supermodular in (\hat{x}, y) on S_j . Lattices, sublattices and supermodularity are defined in Topkis (1998). I now establish that properties (a)-(c) hold.

(a) Let $\hat{X}_j \equiv \{(x, \underline{x}) : x \geq \underline{x}_j^k \text{ for } k = 1, 2 \text{ and } \underline{x} \in \{\underline{x}^1, \underline{x}^2\}\}$. Here \underline{x}^1 and \underline{x}^2 satisfy $\underline{x}_j^2 \geq \underline{x}_j^1, \forall j$. Order points with the coordinate order (i.e. $(x, \underline{x}) \geq (x', \underline{x}')$ provided $x \geq x'$ and $\underline{x}_j \geq \underline{x}'_j, \forall j$). \hat{X}_j is a lattice since the pairwise sup and inf operations are defined and the resulting points lie in \hat{X}_j . $Y = R^1$ is a lattice with the usual order.

(b) $(\hat{x}, y), (\hat{x}', y') \in S_j$ implies that $\text{sup}((\hat{x}, y), (\hat{x}', y')) \in S_j$ for two reasons. First, $\text{sup}(\hat{x}, \hat{x}') \in \hat{X}_j$ as \hat{X}_j is a lattice. Second, $\text{sup}(\underline{s}_{j+1}, \underline{s}'_{j+1}) \leq \text{sup}(y, y') \leq \text{sup}(x, x')$. A parallel argument establishes that the same holds for the inf. Thus, S_j is a sublattice.

(c) $f_j(\hat{x}, y)$ is supermodular in (\hat{x}, y) if each of the two component functions is supermodular in (\hat{x}, y) . The first component $u(x - y)$ is supermodular in (\hat{x}, y) as $u(x - y)$ is supermodular in (x, y) by Topkis (1998, Lemma 2.6.2) and the extension to (\hat{x}, y) clearly holds. It remains to show that $E[V_{j+1}(y(1+r) + z'; \underline{x})]$ is supermodular in (\hat{x}, y) for all j . This follows from backwards induction using steps 1-4.

Step 1: $V_{J+1} \equiv 0$. Thus, $V_{J+1}(x; \underline{x})$ is supermodular in $\hat{x} = (x, \underline{x})$.

Step 2: $V_{j+1}(x; \underline{x})$ supermodular in $\hat{x} = (x, \underline{x})$ implies that $V_{j+1}(y(1+r) + z'; \underline{x})$ is supermodular in (\hat{x}, y) , given z' .

Step 3: $V_{j+1}(y(1+r) + z'; \underline{x})$ supermodular in (\hat{x}, y) implies that $E[V_{j+1}(y(1+r) + z'; \underline{x})]$ is supermodular in (\hat{x}, y) by Topkis (1998, Corr. 2.6.2).

Step 4: Topkis (1998, Thm. 2.7.6) implies that $V_j(x; \underline{x})$ is supermodular in \hat{x} since $f_j(\hat{x}, y)$ is supermodular in (\hat{x}, y) by steps 1-3 and since assumptions (a)-(b) hold. This result can be loosely paraphrased as saying that the maximum of a supermodular function is supermodular over the remaining unmaximized variables. \square

Proof of Lemma 1 (iii):

Proceed by induction. $s_j(x; \theta)$ is increasing in θ for $j = J$ as $s_J(x; \theta) = 0$ since no borrowing is allowed in the last period of life. Given that this holds for j show that it holds for $j - 1$. As a first result note that the inequalities below hold, given $\theta \geq_{\Theta} \theta'$. The leftmost inequality holds when V'_j is convex in x . Schechtman (1976, Thm. 1.10) and Miller (1976, Lemma 1) prove that V'_j is convex in x when u'_j is convex. The rightmost inequality holds when V'_j is increasing in θ . This holds by the induction hypothesis since $V'_j(x; \theta) = u'(x - s_j(x; \theta))$.

$$E_{\theta}[V'_j(s'(1+r) + z'; \theta)] \geq E_{\theta'}[V'_j(s'(1+r) + z'; \theta)] \geq E_{\theta'}[V'_j(s'(1+r) + z'; \theta')]$$

Next suppose by contradiction that $s_{j-1}(x; \theta) < s_{j-1}(x; \theta')$ for some x , given $\theta \geq_{\Theta} \theta'$. In the equation below the topmost inequality holds by the Euler equation. The bottommost inequality follows from the result above and the fact that V'_j is decreasing

in wealth since V_j is concave in wealth. The equality holds by the Euler equation since $s_{j-1}(x; \theta')$ is interior.

$$\begin{aligned} u'(x - s_{j-1}(x; \theta)) &\geq \beta(1+r)E_\theta[V'_j(s_{j-1}(x; \theta)(1+r) + z'; \theta)] \\ &\geq \beta(1+r)E_{\theta'}[V'_j(s_{j-1}(x; \theta')(1+r) + z'; \theta')] = u'(x - s_{j-1}(x; \theta')) \end{aligned}$$

A contradiction follows as $s_{j-1}(x; \theta) < s_{j-1}(x; \theta')$ and u strictly concave imply that the marginal utility of consumption must be strictly smaller under $s_{j-1}(x; \theta)$ than under $s_{j-1}(x; \theta')$. This completes the induction. \square

The proof of Lemma 1(iv) rests on Lemma 2 below, which is a restatement of Carroll and Kimball (2001, Lemma 1-2). The proof of Lemma 2 is essentially identical to that in Carroll and Kimball (2001) with the exception that the proof of Lemma 2(ii) is shorter, simpler and more general than that appearing in an early version of their paper. The proof of Lemma 1(iv) uses the following definition. A differentiable function $F(x)$ has property CC with respect to u provided $F'(x) = u'(f(x))$ for some increasing, concave function $f(x)$. The notation for the earnings process θ is dropped for convenience throughout the proof.

Proof of Lemma 1 (iv):

$V_J(x)$ has property CC with respect to u since $V_J(x) = u(x)$. Backwards induction establishes via Lemma 2 that $V_j(x)$ has property CC with respect to u for all j . This property and the fact that $V'_j(x) = u'(c_j(x))$ together imply that $c_j(x)$ is concave in x and that $s_j(x) = x - c_j(x)$ is convex in x for all j . \square

Lemma 2: Assume B1-3.

- (i) $V_{j+1}(x)$ has property CC with respect to u implies $\Omega_j(s) \equiv \beta E[V_{j+1}(s(1+r) + z)]$ has property CC with respect to u .
- (ii) $\Omega_j(s) \equiv \beta E[V_{j+1}(s(1+r) + z)]$ has property CC with respect to u implies $V_j(x)$ has property CC with respect to u .

Proof:

(i) Consider the case of CRRA utility. Assume that $\beta = (1+r) = 1$ for expositional simplicity only. With this simplification, $\Omega'_j(s) = E[V'_{j+1}(s+z)] = E[u'(f(s+z))] = E[f(s+z)^{-\sigma}]$ for some increasing, concave f since V_{j+1} has property CC. It remains to show that $(\Omega'_j(s))^{-1/\sigma}$ is increasing and concave. As this is clearly increasing, it remains to establish concavity. The following three inequalities then hold. The first follows for any z from the monotonicity and concavity of f when $s = ps_1 + (1-p)s_2$ with $p \in [0, 1]$. The second follows from the first. The third is an application of

Minkowski's inequality to the term on the right-hand side of the second inequality (see Hardy et al (1967, Thm. 198)). This requires that $\sigma > 0$ and that both $pf(s_1 + z)$ and $(1 - p)f(s_2 + z)$ are positive random variables.

$$f(s + z) \geq pf(s_1 + z) + (1 - p)f(s_2 + z)$$

$$(E[f(s + z)^{-\sigma}]^{-1/\sigma} \geq (E[(pf(s_1 + z) + (1 - p)f(s_2 + z))^{-\sigma}]^{-1/\sigma}$$

$$(E[(pf(s_1 + z) + (1 - p)f(s_2 + z))^{-\sigma}]^{-1/\sigma} \geq (E[(pf(s_1 + z))^{-\sigma}]^{-1/\sigma} + (E[((1 - p)f(s_2 + z))^{-\sigma}]^{-1/\sigma}$$

Combining these results one obtains the inequality below. This inequality establishes that $(\Omega'_j(s))^{-1/\sigma}$ is concave and, thus, $\Omega_j(s)$ exhibits property CC.

$$(\Omega'_j(s))^{-1/\sigma} \geq p(\Omega'_j(s_1))^{-1/\sigma} + (1 - p)(\Omega'_j(s_2))^{-1/\sigma}$$

Now consider the case of CARA utility. Assume that $\beta = (1 + r) = 1$ for expositional simplicity only. With this simplification, $\Omega'_j(s) = E[V'_{j+1}(s + z)] = E[e^{-af(s+z)}]$ for some increasing, concave f . It is sufficient to show that $-(1/a)\log\Omega'_j(s)$ or $-\log\Omega'_j(s)$ is increasing and concave. As both are clearly increasing, it remains to establish concavity. The first inequality below holds by the monotonicity and concavity of f when $s = ps_1 + (1 - p)s_2$ with $p \in [0, 1]$ and $a > 0$. The second follows from the first. The third holds by applying the arithmetic-geometric mean inequality to the term on the right-hand side of the second inequality, omitting the log. More precisely, the arithmetic-geometric mean inequality implies that for positive random variables m, n with means \bar{m}, \bar{n} , $E[(m/\bar{m})^p(n/\bar{n})^{1-p}] \leq E[p(m/\bar{m}) + (1 - p)(n/\bar{n})] = 1$ and, thus, $E[m^p n^{1-p}] \leq \bar{m}^p \bar{n}^{1-p}$.

$$-af(s + z) \leq -paf(s_1 + z) - (1 - p)af(s_2 + z)$$

$$\log E[e^{-af(s+z)}] \leq \log E[e^{-paf(s_1+z) - (1-p)af(s_2+z)}]$$

$$E[e^{-paf(s_1+z) - (1-p)af(s_2+z)}] \leq E[e^{-af(s_1+z)}]^p E[e^{-af(s_2+z)}]^{(1-p)}$$

Taking the log of both sides of the third inequality above and combining this with the second inequality one obtains the inequality below. This inequality establishes that $\log\Omega'_j(s)$ is convex, $-\log\Omega'_j(s)$ is concave and, thus, $\Omega_j(a)$ exhibits property CC.

$$\log E[e^{-af(s+z)}] \leq p \log E[e^{-af(s_1+z)}] + (1 - p) \log E[e^{-af(s_2+z)}]$$

(ii) Since $V_j(x) = \max_{s' \in \Gamma_j(x)} u(x - s') + \Omega_j(s')$, $V_j'(x) = u'(c_j(x))$ and since $c_j(x)$ is increasing via the concavity of $V_j(x)$, it is sufficient to show that $c_j(x)$ is concave. This occurs if and only if the set $\Lambda_j \equiv \{(c, x) : u'(c) \geq \Omega_j'(x - c); c \geq 0, x - c \geq \underline{s}_{j+1}, x \geq \underline{x}_j\}$ is convex. Since $\Omega_j'(x - c) = u'(f(x - c))$ and u' is a decreasing function, $\Lambda_j = \{(c, x) : c \leq f(x - c); c \geq 0, x - c \geq \underline{s}_{j+1}, x \geq \underline{x}_j\}$. Λ_j convex then follows from the fact that f is increasing and concave using Rockafellar (1970 Thm. 4.6). \square

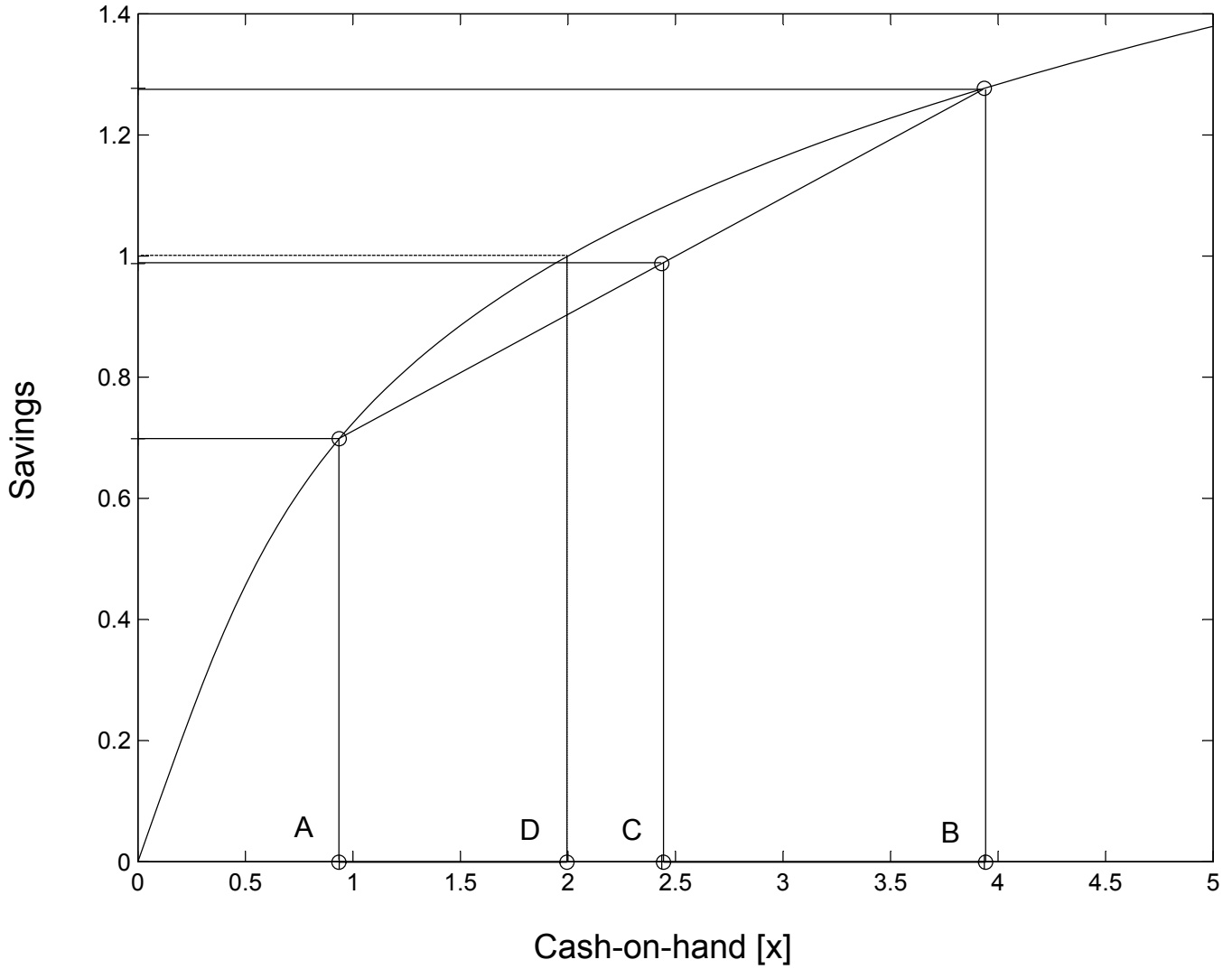
Table 1: Solution

Period	Earnings	Wealth	Cash-on-Hand	Consumption	Expected Wealth
j	z_j	s_j	$s_j(1+r) + z_j$	c_j	$E[s_j]$
1	1.5	0.0	1.5	1.0	0.0
2	1.5	0.5	2.0	1.0	0.5
3	0.0	1.0	1.0	1.0	1.0
Period	Earnings	Wealth	Cash-on-Hand	Consumption	Expected Wealth
j	z_j	s_j	$s_j(1+r) + z_j$	c_j	$E[s_j]$
1	1.5	0.0	1.5	0.5638	0.0
2	0.0	0.9362	0.9362	0.2379	0.9362
	3.0	0.9362	3.9362	2.6592	-
3	0.0	0.6984	0.6984	0.6984	0.9877
	0.0	1.2770	1.2770	1.2770	-

Figure 1: Expected Wealth Profiles



Figure 2: Savings Function at Age 2



A: low state with earnings risk; B: high state with earnings risk
C: mean value of state with earnings risk; D: state without risk