

Aggregate Precautionary Savings: When is the Third Derivative Irrelevant?

Mark Huggett and Sandra Ospina*

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Abstract

When are aggregate precautionary savings positive? We address this question in the context of a general equilibrium model where infinitely-lived agents receive idiosyncratic labor endowment shocks, hold a risk-free asset to smooth consumption and face a liquidity constraint. We prove that (1) the steady-state capital stock is always larger in any equilibrium with idiosyncratic shocks and a liquidity constraint than without idiosyncratic shocks (i.e. aggregate precautionary savings are positive) as long as consumers are risk averse and (2) aggregate precautionary savings occurs if and only if the liquidity constraint binds for some agents.

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*Affiliation: Georgetown University (both authors)

Corresponding Author: Mark Huggett

Address: Economics Department, Georgetown University, Washington DC 20057- 1036

Phone: (202) 687- 6683

Fax: (202) 687- 6102

E-mail: mh5@georgetown.edu

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1 Introduction

It is widely conjectured that uninsured earnings uncertainty may be an important source of aggregate wealth accumulation. This conjecture is supported by household surveys that find that “emergencies” are a popular self-reported reason for saving.¹ It is also supported by some of the empirical work surveyed by Browning and Lusardi (1996) which examines whether household wealth accumulation in panel data is positively related to measures of household earnings uncertainty. Finally, it is implied by simulations of the leading models of consumption and savings behavior.²

This motivates us to ask when is there more aggregate wealth in the presence of uninsured, idiosyncratic earnings uncertainty than in the absence, aggregate labor endowment held constant? We note two things. First, although the question is fundamental, there is little existing theoretical work answering it. Second, it is widely believed that the answer is known. In particular, it is believed that the answer is that this occurs when individuals are not only risk averse but also when the third derivative of the period utility function is positive.

The importance of the third derivative originates from the work of Leland (1968) and Sandmo (1970). They pose the following question. When does an expected utility maximizing individual save more in the presence of uninsured, earnings uncertainty than in its absence, mean earnings held constant? They answer this question in the context of a two-period, partial-equilibrium model in which there is earnings uncertainty in the second period and in which there is only a single, risk-free asset that can be accumulated. They prove that a risk-averse individual will save more in the presence of uncertainty as long as the third derivative of the period utility function is positive. More precisely, they prove that the optimal decision rule for wealth accumulation is increasing in uncertainty.

We believe that the question of when aggregate precautionary savings is positive ought to be analyzed within the context of the dominant frameworks used for both theoretical and empirical investigations of aggregate capital accumulation. These are the overlapping generations and the infinitely-lived agent frameworks. This belief seems to be uncontroversial insofar as the main attempts to quantify the potential magnitudes of aggregate precautionary savings (i.e. Skinner (1988), Caballero (1991), Aiyagari (1994) and Huggett (1996)) have been based on these frameworks. However, these frameworks would seem to allow aggregate wealth accumulation to be determined by distinct considerations than those considered by Leland and Sandmo. In particular, the analysis of wealth accumulation in either of these frameworks is done by comparing steady states. Thus, any such analysis will have to employ the underlying logic of

¹See Projector and Weiss (1966, Table A30).

²See Skinner (1988), Caballero (1991), Aiyagari (1994) and Huggett (1996).

steady-state analysis.

We investigate aggregate precautionary savings within the infinitely-lived agent framework, while Huggett (2000) analyzes this issue within the overlapping generations framework. In our model there are a continuum of infinitely-lived agents that experience idiosyncratic shocks to labor endowment. Paralleling the analysis of Leland and Sandmo, we assume that labor endowment is inelastically supplied and that agents can hold only a risk-free asset. While Leland and Sandmo abstract from the possibility that agents might face a borrowing limit that could be endogenously binding, we do not. Thus, in our model an agent faces a fixed limit in the amount that can be borrowed which we refer to as a liquidity constraint.³ In the aggregate, agents can accumulate physical capital. We will say that there is (positive) aggregate precautionary savings provided that the steady-state capital stock is larger in the presence than in the absence of idiosyncratic labor endowment uncertainty, holding aggregate labor endowment constant.⁴

Theorem 1 of our paper states that the capital stock is larger in any steady-state equilibrium with idiosyncratic uncertainty than without. Thus, aggregate precautionary savings is a necessary feature of a steady-state equilibrium where risk averse agents experience idiosyncratic earnings uncertainty and face a liquidity constraint. This theorem holds regardless of the third or higher order derivative properties of the period utility function, the amount of idiosyncratic uncertainty, the precise level of the liquidity constraint or whether agents differ in preferences and endowment processes.

Theorem 2 of our paper provides a complete characterization of aggregate precautionary savings in the presence of a liquidity constraint. The theorem states that aggregate precautionary savings occurs if and only if the liquidity constraint binds for a positive mass of agents. Thus, our results emphasize the role of endogenously binding liquidity constraints.

We now provide a preview for our results. We find that many economists whose focus is on the precautionary savings literature are surprised that third derivatives are irrelevant for our results. In contrast, economists whose focus is on capital theory are not surprised. Capital theorists are not surprised as steady states in models with infinitely-lived agents can be characterized directly from the Euler equation using only the assumption that marginal utility is diminishing in consumption. In particular,

³Section 3 of the paper specifies more precisely what we mean by a liquidity constraint. We note that the presence of borrowing limits in dynamic models is not really the issue. What matters is the exact form of the borrowing limits that agents face. We expand upon this issue in section 5.

⁴We note that this steady-state notion of aggregate precautionary savings is the standard one in the literature (see Skinner (1988), Caballero (1991), Aiyagari (1994) and Huggett (1996)). An alternative notion would compare paths for aggregate capital in the presence and absence of idiosyncratic uncertainty, starting from the same initial distribution of capital and holding the path of aggregate labor endowment constant.

the fact that in a steady state the distribution of marginal utility cannot change over periods allows one to pin down the steady-state capital stock in one-sector models. This paper shows that the same reasoning, again using only diminishing marginal utility, also applies when characterizing steady states in similar models with idiosyncratic uncertainty.

The paper is organized in six sections. Section 2 describes how our paper fits into the literature. Section 3 describes the model economy. Section 4 presents the main results as well as the logic used to prove these results. Section 5 discusses our results. Section 6 concludes.

2 Literature Review

We review the theoretical literature that bears on the question of when aggregate wealth holding is larger with idiosyncratic earnings uncertainty than without. Our review is divided into three parts. Parts one and two focus on results relevant for overlapping generations and infinitely-lived agent models respectively. The third part presents some contributions to the literature which are relevant but do not otherwise fall naturally into the first two categories.

2.1 Overlapping Generations Economies

The early literature on precautionary savings is associated with Leland (1968), Sandmo (1970) and the other authors listed in Table 1. They focused on a two-period model where future earnings z_2 are random and are drawn from a distribution indexed by the parameter θ . Agents maximize an expected utility function $E[u_1(c_1) + u_2(c_2)]$ by choosing the amount of a risk-free asset to carry to the next period.⁵ As indicated in Table 1, the main result is that the optimal decision rule has the property that wealth holding $k(x, j; \theta)$ carried to the next period increases in uncertainty θ from any initial state x and age j .⁶ A state $x = (k, z)$ consists of wealth holding k brought into the period and earnings z . The result follows from the Euler equation below, which ignores corner solutions for simplicity. Here r denotes the interest rate.

$$u'_1(k_1(1+r) + z_1 - k_2) = E[u'_2(k_2(1+r) + z_2)](1+r)$$

The intuition for the result is that with convex marginal utility (u'_2 convex or alternatively $u''_2 > 0$) increases in earnings uncertainty increase expected future marginal

⁵We focus on additive utility in order to facilitate comparisons with the rest of the literature.

⁶Increasing uncertainty is defined as in Rothschild and Stiglitz (1970). The Rothschild-Stiglitz definition induces a partial order on alternative values of θ , where θ indexes earnings distributions.

utility of consumption for any fixed level of wealth k_2 carried to the second period. Thus, as uncertainty increases it is optimal to carry more wealth to the second period. The subsequent literature associated with Miller (1975, 1976), Sibley (1975), Schechtman (1976) and Mendelson and Amihud (1982) generalizes this result to apply to models where preferences are $E[\sum_{j=1}^J u_j(c_j)]$, earnings shocks are independent and there are arbitrarily many model periods J . The intuition for the result is the same.

[Insert Table 1 Here]

The results summarized above do not imply that aggregate wealth holding increases with increases in uncertainty. Huggett (2000) argues that the key result which is sufficient to imply that aggregate wealth holding increases with uncertainty is that the expected wealth holding profile over an agent's life cycle increase with uncertainty.⁷ Such a profile is displayed in Figure 1 for one earnings process indexed by θ' and for a riskier earnings process indexed by θ .

[Insert Figure 1 Here]

Within the context of Figure 1, imagine that we have an economy where all agents at birth start out with the same wealth, experience draws from the same earnings distribution and employ the same decision rule to determine wealth holding carried into future periods. Here we assume that earnings draws are independent over time as well as across agents. In this economy aggregate wealth holding is a weighted sum of the average wealth holding of agents of different ages that are alive at the same point in time. By the law of large numbers, the average wealth holding of a large cohort of agents of a given age equals, with probability one, the expected wealth holding displayed in Figure 1. For this reason, if the entire expected wealth holding profile shifts upwards as uncertainty increases, then aggregate wealth holding also increases with uncertainty in partial equilibrium.

Under the conditions described above, Huggett (2000) shows that when a decision rule for wealth accumulation has three key properties then the expected wealth profile in Figure 1 shifts upwards as uncertainty increases. These key properties are that the decision rule $k(x, j; \theta)$ is increasing in uncertainty θ and is increasing and convex in the state x . Furthermore, Huggett (2000) argues that if any of these three key properties are dropped, then counter examples can be found where the pattern in Figure 1 does not hold. These counter examples can be found in models where agents live more than two model periods.⁸ Thus, increasing aggregate wealth holding in overlapping

⁷This result, just like the other results in Table 1, is a partial equilibrium result.

⁸In models where agents live for precisely two periods the Leland-Sandmo result is sufficient for aggregate wealth holding to increase with uncertainty.

generations economies depends, even in a partial equilibrium context, on properties beyond those stressed in the work of Leland, Sandmo and the other authors cited in Table 1.

2.2 Infinitely-Lived Agent Economies

There is an extensive literature which analyzes the partial equilibrium problem of optimal consumption and savings behavior when an agent lives forever, experiences independent earnings shocks and can hold only a risk-free asset. An agent maximizes an expected utility function $E[\sum_{j=0}^{\infty} \beta^j u(c_j)]$, where $\beta = 1/(1 + \delta)$ is a discount factor and $\delta > 0$ is a time preference rate. Some of the important results which are relevant to the issue at hand are summarized in Table 2. In particular, Mendelson and Amihud (1982, Theorem 6.1) and Sotomayor (1984, Theorem 2.8 and 4.5)) prove that an agent's asset accumulation becomes infinite over time almost surely when the interest rate r equals or exceeds the time preference rate δ .⁹ We note that this result is closely related to our Theorem 1. Our Theorem 1 states that a steady state with an interest rate greater than or equal to the time preference rate is impossible when there is idiosyncratic earnings uncertainty and a liquidity constraint. Stated somewhat differently, we prove that the capital stock in any steady-state equilibrium with idiosyncratic uncertainty is larger than the capital stock that would obtain without idiosyncratic uncertainty and thus the interest rate r must be less than the time preference rate δ .

[Insert Table 2 Here]

Theorem 1 in our paper is related to the papers by Aiyagari (1995) and Huggett (1997). Both authors prove that for the case of idiosyncratic uncertainty that in steady state $r < \delta$ or, equivalently, that $\beta(1 + r) < 1$. We note two things. First, in contrast to the present paper, neither of these papers focus directly on the issue of when aggregate precautionary savings occurs.¹⁰ Second, the result in our paper has a number of advantages over that in Aiyagari. In particular, our paper (i) makes it clear that the result is due neither to imposing a borrowing limit of zero (i.e. no borrowing at all) nor to imposing an Inada condition which makes the marginal utility of additional consumption infinite at zero consumption, (ii) employs a logic which permits a short and simple proof based on first principles that also extends naturally to cover the case of Markov shocks and (iii) leads immediately to the insight, contained in Theorem 2

⁹Chamberlain and Wilson (1984) prove a closely related result.

¹⁰Aiyagari (1994) does focus on this issue. He presents computational results on the magnitude of aggregate precautionary savings rather than theoretical results on when this occurs.

of our paper, that aggregate precautionary savings is equivalent to the phenomena of endogenously binding liquidity constraints.

Another important result from the literature focuses on the existence of steady-state equilibria. The papers by Laitner (1979) and Clarida (1990) prove that there exists an equilibrium where the interest rate r is less than the time preference rate δ .¹¹ An equivalent statement is that there exists a steady-state equilibrium where $\beta(1+r) < 1$ when there is idiosyncratic uncertainty. Their argument is based on showing that in partial equilibrium aggregate wealth accumulation by agents becomes infinite as the interest rate approaches the time preference rate from below. This result and the continuity of the relationship between aggregate asset accumulation and the interest rate imply that there is at least one interest rate for which steady-state capital supply equals the capital demanded by competitive firms. This result, while quite suggestive, leaves open the issue of whether there are other equilibria for which $r \geq \delta$ or $\beta(1+r) \geq 1$. Our results show very generally that equilibria with $\beta(1+r) \geq 1$ are not possible and that this has nothing to do with third derivative conditions.

2.3 Other Contributions

We mention two other strands of the literature on the topic of aggregate precautionary savings. First, the papers by Skinner (1988), Caballero (1991), Aiyagari (1994) and Huggett (1996) all attempt to quantify the potential magnitudes of aggregate wealth accumulation due to uninsured earnings uncertainty. Each author demonstrates that earnings uncertainty accounts for a positive fraction of aggregate wealth accumulation. These papers all employ specific functional forms for preferences and for earnings processes. In fact, all these papers employ preferences where the third derivative of the utility function is positive. Caballero's result is based on finding an analytic solution to the problem posed (the guess and verify method), whereas the other results use computational methods to approximate the solution to the problems studied. Second, Deaton (1991, 1992) considers the consumption and saving behavior of an infinitely-lived agent who faces the problem studied in the previous subsection. The discussion in Deaton (1992, CH. 6 and especially pp. 197-98) is suggestive of what features of this model may determine how aggregate wealth accumulation is related to earnings uncertainty. Our paper advances the debate about what these determinants are by providing precisely stated Theorems bearing on this issue.

¹¹Clarida's paper makes use of a proposition, which Clarida attributes to Bewley (1984), stating that aggregate wealth accumulation by agents is continuous as a function of the interest rate r , when $r < \delta$.

3 The Economy

We describe the economy in three steps. First, we state our assumptions. Second, we describe the decision problem of agents and state some necessary conditions of optimal decisions. Third, we define our steady-state equilibrium concept.

3.1 Assumptions

There are a continuum of infinitely-lived agents in the economy. The total mass of agents is equal to 1. All agents have identical preferences over consumption that are given by a calculation of expected utility:

$$E\left[\sum_{j=0}^{\infty} \beta^j u(c_j)\right]$$

Assumptions A1-2 list the restrictions on the period utility function u and the discount factor β . Assumption A1 ensures that expected utility will be bounded. Assumption A2 lists some standard properties of the period utility function u .

A1 $u(c)$ is bounded and $\beta \in (0, 1)$.

A2 u is strictly increasing, strictly concave and continuously differentiable.

Each period each agent receives a labor endowment $e \in E$ measured in efficiency units. Assumption A3 says that in any time period the labor endowment is potentially uncertain. This uncertainty is represented mathematically by a probability measure P defined on \mathcal{E} , the Borel sets of E . We will say that there is *idiosyncratic uncertainty* when $P(e) \neq 1$ for any $e \in E$, whereas there is *no idiosyncratic uncertainty* when $P(e) = 1$ for some $e \in E$. Assumption A3 also says that the expected labor endowment of any agent is finite and normalized to equal 1. Here it is understood that labor endowments are independent and identically distributed for a given agent over time as well as independent across agents. Thus, there is potentially uncertainty for an individual agent but no uncertainty over the aggregate labor endowment which always equals 1.

A3 (E, \mathcal{E}, P) is a probability space.

$$E = [\underline{e}, \infty) \subseteq R_+, \mathcal{E} \text{ are the Borel sets of } E \text{ and } \int_E e dP = 1$$

There is a single firm in the economy that operates a total output function f . Assumption A4 lists the restrictions on f . These assumptions are the usual ones employed in neoclassical growth theory and imply that the marginal product of capital is positive and diminishing. Assumption A4 allows for the case of a constant returns

to scale production function $F(K, L)$ in capital K and labor L inputs, where capital depreciates at a constant rate δ . In this case the total output function f is given as follows: $f(K) = F(K, 1) + K(1 - \delta)$.

$$A4 \quad f(0) = 0, \quad f' > 0 \quad \text{and} \quad f'' < 0$$

3.2 Decision Problem

The decision problem that an individual agent faces is a version of the decision problem studied in the theoretical literature that was reviewed in section 2.1. Thus, an agent experiences random variation over time in the efficiency of his/her labor endowment. An agent faces a fixed real wage w per efficiency unit of labor and a fixed real interest rate r at which it can borrow and lend as long as asset holdings stay above a borrowing limit, \underline{k} . An agent maximizes expected utility.

This decision problem is stated below in the language of dynamic programming. The individual state variable $x = (k, e)$ of a particular agent contains the current period values of asset holding k and the labor endowment shock e . The individual state x lies in the individual state space $X = [\underline{k}, \infty) \times E$. Expectations in the dynamic programming problem are taken with respect to the probability measure P defined on endowment shocks. A solution to this problem is given by optimal decision rules for consumption $c(x)$ and next period asset holding $k(x)$ as well as an optimal value function $V(x)$ satisfying the following equation:

$$V(x) = \max_{(c, k') \in \Gamma(x; w, r, \underline{k})} u(c) + \beta E[V(k', e')]$$

The period budget set $\Gamma(x; w, r, \underline{k})$ allows an agent to divide current period resources between consumption c and next period asset holdings k' . Current period resources equal the value of the current period labor endowment ew and the value of asset holdings $k(1 + r)$. In addition, an agent faces a borrowing limit \underline{k} in that next period asset holdings must lie above this level.

$$\Gamma(x; w, r, \underline{k}) = \{(c, k') : c + k' \leq ew + k(1 + r), c \geq 0, k' \geq \underline{k}\}$$

We put two restrictions on the borrowing limit \underline{k} , given that w and $(1 + r) > 0$. First, agents face a liquidity constraint in that the borrowing limit \underline{k} is set at any level above the level associated with solvency. The limit associated with solvency is the level \underline{k} such that when assets are at this level and when the agent receives the lowest endowment shock \underline{e} all resources would be used to maintain assets at the borrowing

limit (i.e. $\underline{k} = \underline{e}w + \underline{k}(1+r)$) and, hence, consumption is zero.¹² Second, \underline{k} must lie below the level of the capital stock per person, K , in the economy. The liquidity constraint restrictions described above are summarized as follows:

$$\underline{k} < \underline{e}w + \underline{k}(1+r) \text{ and } \underline{k} < K$$

3.3 Properties of the Decision Problem

Lemma 1 below lists some properties of optimal value functions $V(x)$ and optimal decision rules $c(x)$ and $k(x)$. We use the necessary condition from Lemma 1 (2)(iii) in each of the main theorems of the paper. This necessary condition is stated in terms of the right-hand derivative of the value function with respect to its first argument, $V_1^+(x)$. It states that the marginal utility gain to additional asset holdings equals the maximum of its use in current consumption or its use in asset accumulation. Lemma 1 (3) establishes that for low enough interest rates there are values of the individual state x such that the borrowing limit will be strictly binding. We will say that the borrowing limit strictly binds when $V_1^+(x) > \beta(1+r)E[V_1^+(k(x), e')]$. This just states that the best use of an increment in assets is for current consumption rather than for asset accumulation. Lemma 1 (4) establishes that for low enough interest rates individual agents will always receive labor endowment shocks which lead them to reduce asset holdings if they were not already at the borrowing limit.

To prove Lemma 1 we use the following standard results for this type of problem. First, a unique, continuous and bounded function $V(x)$ exists that solves the decision problem stated above. Second, V is strictly increasing and strictly concave in asset holdings k , given strictly positive factor prices w and $(1+r)$.

Lemma 1: Assume A1-3, $w > 0$, $(1+r) > 0$ and that agents face a liquidity constraint.

- (1) $c(x)$ and $k(x)$ are continuous.
- (2) (i) $V_1^+(x)$ exists and is strictly decreasing in both arguments.
(ii) $V_1^+(x)$ is bounded
(iii) $V_1^+(x) = \max\{u'(c(x))(1+r), \beta(1+r)E[V_1^+(k(x), e')]\}$
- (3) If there is idiosyncratic uncertainty and $\beta(1+r) \leq 1$, then there exists a set $B = [\underline{k}, a) \times A$ where $a > \underline{k}$ and $P(A) > 0$ such that the borrowing limit strictly binds for all $x \in B$.

¹²When $r \leq 0$ the limit associated with solvency does not exist. In this case we allow any borrowing limit such that $\underline{k} < \underline{e}w + \underline{k}(1+r)$.

(4) If there is idiosyncratic uncertainty and $\beta(1+r) \leq 1$, then for any $k_1 > \underline{k}$ there exists $\epsilon, \delta > 0$ and $A \in \mathcal{E}$ where $P(A) > 0$ such that $k(k, e) \leq k_1 - \epsilon$ for all $k \in [k_1, k_1 + \delta)$ and for all $e \in A$.

Proof: See the Appendix.

In the analysis that follows we make use of the general necessary condition $V_1^+(x) \geq \beta(1+r)E[V_1^+(x')]$ from Lemma 1 (2)(iii) rather than the more familiar necessary condition $u'(c(x)) \geq \beta(1+r)E[u'(c(k(x), e'))]$. This latter condition follows from the former when $V_1^+(x) = u'(c(x))(1+r)$.¹³ We note that $u'(0) = \infty$ is sufficient for $V_1^+(x) = u'(c(x))(1+r)$. As this need not hold when $u'(0) < \infty$, we use the general necessary condition in all of our arguments.

3.4 Equilibrium

In this section we provide a definition of steady-state equilibrium. This equilibrium concept requires that the distribution of agents across individual states x does not change over time even though individual agents can move within the distribution. The distribution is described mathematically by a probability measure λ defined on (X, \mathcal{X}) , where $X = [\underline{k}, \infty) \times E$ is the individual state space and \mathcal{X} are the Borel sets of X .¹⁴ To state the equilibrium concept used in this paper we define a transition function $P(x, B) \equiv P(\{e' : (k(x), e') \in B\})$ on the sets $B \in \mathcal{X}$. The transition function gives the probability that an agent with state x this period will have a state in the set B next period.

Definition: A steady-state equilibrium with a liquidity constraint is $(c(x), k(x), K, w, r, \underline{k}, \lambda)$ such that

(1) $c(x)$ and $k(x)$ are optimal decision rules, given (w, r, \underline{k}) .

(2) *Factor Prices are Competitively Determined:*

$$1+r = f'(K) \text{ and } w = f(K) - f'(K)K$$

(3) *Resource Feasibility:*

$$(i) K = \int_X k d\lambda \text{ (ii) } \int_X e d\lambda = 1 \text{ and (iii) } \int_X (c(x) + k(x)) d\lambda = f(K)$$

(4) *Distributions Are Time Invariant:*

$$\lambda(B) = \int_X P(x, B) d\lambda \text{ for all sets } B \in \mathcal{X}$$

(5) *Liquidity Constraint Restrictions:*

$$\underline{k} < \underline{e}w + \underline{k}(1+r) \text{ and } \underline{k} < K$$

¹³Mirman and Zilcha (1975, Lemma 1) prove that this equality holds when both asset holding and consumption are off corners.

¹⁴The treatment of agent heterogeneity is similar to that in Lucas and Prescott (1974), Foley and Hellwig (1975) and Lucas (1980).

4 Results

The main results of the paper are contained in two theorems. This section states the main results, presents the logic used to prove these results and provides several comments indicating how the results can be extended to apply to economies with heterogeneity in preferences and endowment processes. All other discussion of Theorems 1 and 2 is contained in the next section.

Theorem 1 states that the real interest rate r is always lower in economies with idiosyncratic uncertainty and a liquidity constraint than in economies without idiosyncratic uncertainty. As the steady-state capital stock increases as the interest rate decreases, Theorem 1 states that aggregate precautionary savings is a necessary feature of an equilibrium with idiosyncratic uncertainty and a liquidity constraint.

The logic used to prove Theorem 1 is as follows. Any statistic of the aggregate state of the economy λ cannot change over time in a steady-state equilibrium. Applied to economies with no idiosyncratic uncertainty, this logic implies that $\beta(1+r) = 1$. Otherwise, the marginal utility of consumption when summed over agents or, alternatively, the marginal gain in utility to additional asset holdings when summed over agents would change over time. This is shown by integrating the necessary condition $u'(c(x)) \geq \beta(1+r)E[u'(c(x'))|x]$ or, somewhat more generally, the necessary condition $V_1^+(x) \geq \beta(1+r)E[V_1^+(x')|x]$ using the steady-state distribution λ .¹⁵ In economies with idiosyncratic uncertainty, the same argument shows that $\beta(1+r) \leq 1$. However, in the presence of idiosyncratic uncertainty, a steady state with $\beta(1+r) = 1$ is not possible. The reason is that optimal smoothing behavior implies that agents would draw down asset holdings so that the liquidity constraint would be strictly binding for a positive mass of agents. This follows from Lemma 1 (4) which shows that when $\beta(1+r) \leq 1$ the optimal decision rule for asset holding implies that when agents receive low endowment shocks they choose to reduce asset holding. Reasoning in this way, one concludes that the the marginal gain to additional asset holding when summed over agents must be shrinking over time, since the liquidity constraint would then bind for some agents. This contradiction establishes that $\beta(1+r) = 1$ is not possible.

It should be noted that in all of our results the trivial steady state with zero capital is ignored. Thus, all results implicitly concern steady states with positive capital.

Theorem 1: Assume A1-4.

- (i) *Consider the model economy without idiosyncratic uncertainty. In a steady-state equilibrium with a liquidity constraint $\beta(1+r) = 1$.*

¹⁵See Lemma 1 (2)(iii) and the discussion related to this Lemma to understand how each of these conditions are related.

(ii) Consider the model economy with idiosyncratic uncertainty. In a steady-state equilibrium with a liquidity constraint $\beta(1+r) < 1$.

Proof: See the Appendix.

Comment 1.

Theorem 1 (ii) can easily be extended to economies with heterogeneity in preferences and endowment processes. The reason is that the logic used to prove Theorem 1 (ii) can be applied separately to each of a finite number of agent types. Here an agent type is characterized by their preferences and their endowment processes. When discount factors differ across types, then the result is that with idiosyncratic uncertainty $\beta_i(1+r) < 1$ for the discount factor β_i associated with any agent type i .

Comment 2.

Becker (1980) and Hernandez (1991) characterize steady states in similar models but where agents differ in preferences and endowments and do not experience idiosyncratic uncertainty. They prove that in steady state $\beta(1+r) = 1$, where β is the highest discount factor associated with any agent type. The argument used in Theorem 1 (i) could easily be modified to handle economies with preference and endowment heterogeneity and thus to offer a different proof of this result.

The second result of the paper provides a complete characterization of aggregate precautionary savings in the presence of a liquidity constraint. Theorem 2 states that aggregate precautionary savings occurs if and only if the liquidity constraint is endogenously binding for a positive mass of agents. Theorems 1 and 2 then jointly imply that binding liquidity constraints are a necessary feature of any equilibrium with idiosyncratic uncertainty and a liquidity constraint. The reader will recall that the liquidity constraint is binding for an agent when $V_1^+(x) > \beta(1+r)E[V_1^+(k(x), e')]$ and is not binding when $V_1^+(x) = \beta(1+r)E[V_1^+(k(x), e')]$.

The logic used to prove Theorem 2 is quite simple. The logic is that the condition for aggregate precautionary savings ($\beta(1+r) < 1$) is precisely the condition that the liquidity constraint binds for a positive mass of agents in steady state. This follows from integrating the above necessary conditions over the population using the steady-state distribution λ .

Theorem 2: Assume A1-4. Consider the model economy with idiosyncratic uncertainty. In a steady-state equilibrium with a liquidity constraint, $\beta(1+r) < 1$ if and only if the liquidity constraint binds for a positive mass of agents.

Proof: See the Appendix.

Comment 3.

Theorem 2 can be restated to apply to economies with a positive mass of each of a finite number of agent types, where type refers to both preferences and endowment processes as discussed in Comment 1. The theorem could then be stated as follows: $\beta_i(1+r) < 1$ for each agent type i if and only if the liquidity constraint binds for a positive mass of each agent type. The proof is exactly the same.

5 Discussion

In this section we discuss which features of the model economy are critical for producing aggregate precautionary savings. Our discussion takes the form of answers to several natural questions: (1) Is a borrowing limit key?, (2) Are infinitely-lived agents key? and (3) Is a general equilibrium model key?

Is a Borrowing Limit Key?

It is clear that there must be borrowing limits in any meaningful dynamic model of individual decision making or else all consumption plans lie in the budget set. So the interesting issue is what type of borrowing limits might one want to consider rather than what would happen in the absence of borrowing limits. Given that the point of the precautionary savings literature is to investigate consumer behavior in the presence of uninsurable, idiosyncratic uncertainty, it would not make sense to consider borrowing limits that depend on an individual's shock history in such a way so as to effectively offer complete insurance. This would simply make aggregate precautionary savings zero by construction. Furthermore, as the environment facing an agent is stationary, it seems natural to focus on borrowing limits that are independent of time as well as history. Lastly, given that the literature has focused on the accumulation of risk-free assets it makes sense to focus on borrowing limits that are consistent with these assets being risk free. For these reasons we focus on borrowing limits that are independent of time and shock history (i.e. $k' \geq \underline{k}$) and that make these assets riskless.

From this perspective, the interesting question is not 'Is a borrowing limit key?' but rather 'What happens at different values of \underline{k} ?'. Theorem 1 of this paper answers this question for all values of \underline{k} that satisfy the condition $\underline{k} < \underline{e}w + \underline{k}(1+r)$. This case has been termed the case of a liquidity constraint. This case covers all borrowing limits of interest except the extreme case of a solvency constraint discussed in section 3.2. A solvency constraint allows agents to borrow up to the point where consumption could potentially be zero when an agent receives the worst earnings shock and is already at the borrowing limit: $\underline{k} = \underline{e}w + \underline{k}(1+r)$. For example, $\underline{k} = 0$ constitutes a solvency constraint when agents receive zero earnings with positive probability.

We do not have theoretical results for the solvency constraint case. However, Krusell and Smith (1998) report computational results suggesting that there is aggregate precautionary savings in the particular economies with a solvency constraint that they have considered.¹⁶

Are Infinitely-Lived Agents Key?

Given the well-known equivalence of an infinitely-lived agent with a sequence of finitely-lived agents comprising the generations of an altruistically linked family, it is clear that the results of this paper also apply to at least some situations where agents live for finitely many periods. Thus, a narrow answer to the question is that infinite lifetimes are not absolutely fundamental to our results.

A more interesting question would ask when aggregate precautionary savings occurs in models with finitely-lived agents having no altruistic links. This is the focus of the overlapping generations framework. One can easily construct examples with overlapping generations of two-period-lived agents with quadratic preferences where the addition of labor endowment uncertainty in the second period of life (holding second period mean endowment constant for each agent) leaves unchanged all steady-state aggregates. This works as long as all agents are interior on their savings decision (i.e. borrowing limits do not bind) before and after the introduction of idiosyncratic uncertainty. Thus, with finite lifetimes agents need not optimally choose to hit their borrowing limits for at least some labor endowment processes. This is a key difference with infinitely-lived agent models with the stationary labor income processes considered here. In these models optimal smoothing dictates that agents eventually hit their borrowing limits for any settings of this limit corresponding to a liquidity constraint.¹⁷

In summary, aggregate precautionary savings is not a necessary feature of an equilibrium with idiosyncratic uncertainty in at least some overlapping generations models. Thus, a broader answer to the question that this section addresses would be that finite lifetimes and a lack of altruistic links across generations can be key.

Is a General Equilibrium Model Key?

To address this question one can focus on analyzing aggregate precautionary savings in a partial equilibrium context where factor prices are set at strictly positive but

¹⁶It is easy to show that $\beta(1+r) = 1$ in the case of no idiosyncratic uncertainty and a solvency constraint. In the case of idiosyncratic uncertainty and a solvency constraint the integral that we have focused on, $\int_X V_1^+(x)d\lambda$, may be infinite. If infinite, then our approach is silent on the value of the interest rate. If finite and $u'(0) = \infty$, then $\beta(1+r) = 1$ and there is no aggregate precautionary savings. We leave this issue for future research.

¹⁷See Lemma 1 (3-4) and the section of the proof of Theorem 1 showing that $\beta(1+r) \neq 1$.

arbitrary values. As the results of this analysis are independent of the precise level of the real wage, there are then three cases to consider: Case 1: $\beta(1+r) < 1$, Case 2: $\beta(1+r) = 1$ and Case 3: $\beta(1+r) > 1$.

In case 1 it is clear that all agents are eventually at the borrowing limit (i.e. $k = \underline{k}$) when there is no idiosyncratic uncertainty. The aggregate net capital owned by these agents is therefore $K = \underline{k}$. With idiosyncratic uncertainty it is clear that aggregate net capital must be at least as large, holding the borrowing limit constant. In fact, aggregate net capital will be strictly larger under two circumstances. First, a given borrowing limit may no longer be feasible in the presence of uncertainty. In this case aggregate capital must be strictly larger simply because uncertainty acts to raise the set of feasible borrowing limits. Second, aggregate capital will be strictly larger, holding the borrowing limit equal, in any situation in which smoothing consumption is beneficial. This is the case as agents can only smooth consumption by holding assets.

In case 2 any distribution of capital across agents is a steady-state distribution when agents face no idiosyncratic uncertainty. In this case it is always optimal to consume the interest on asset holdings. The value of K is therefore indeterminate. When agents face idiosyncratic uncertainty the results of Sotomayor (1984, Theorems 2.8 and 4.5) guarantee that when $\beta(1+r) \geq 1$ the asset holdings of any individual will go to infinity with probability 1. This result suggests that no steady state distribution would exist with idiosyncratic uncertainty. In fact, this is precisely what was shown in the proof of Theorem 1 for any interest rate satisfying $\beta(1+r) \geq 1$. Thus, there is no theoretical result on aggregate precautionary savings, as we have defined it, as steady states do not exist when $\beta(1+r) = 1$.

One can now see that case 3 runs into the same incomparability problem as in case 2. However, in case 3 the problem is even more severe as steady states do not exist even in the absence of idiosyncratic uncertainty. Hence, the only way to talk about aggregate precautionary savings is to make path comparisons. This is beyond the scope of this paper.

In summary, the only situation that permits steady-state comparisons of idiosyncratic and no idiosyncratic uncertainty is the case where $\beta(1+r) < 1$. For these interest rates the partial equilibrium comparisons yield a similar conclusion as was previously obtained in Theorem 1. This conclusion is that there is at least as much capital in a steady state with idiosyncratic uncertainty than without whenever steady states can be compared.

6 Conclusion

The conventional wisdom among many economists is that precautionary savings is theoretically related to third derivative properties of expected utility representations of preferences. This paper focuses on the question of when aggregate precautionary savings occurs in economies populated by infinitely-lived agents who face earnings uncertainty and a liquidity constraint. Within this context it has been shown that (1) aggregate precautionary savings is a necessary implication of any equilibrium where risk-averse agents experience idiosyncratic uncertainty and face a liquidity constraint and (2) aggregate precautionary savings coincides with the phenomena of endogenously binding liquidity constraints. Thus, third derivative conditions are irrelevant for the issue of when aggregate precautionary savings occurs within this model in the precise sense that any steady-state capital stock is larger with idiosyncratic uncertainty than without, regardless of the sign of the third derivative.

These results have been examined for robustness. We have shown that they hold for any amount of idiosyncratic uncertainty as well as any value of the liquidity constraint that is feasible in equilibrium. These results hold independent of third or higher order derivative conditions, independent of the rate at which agents discount future utility and independent of whether or not there are several types of agents in the economy differing in preferences and endowment processes. However, the results do depend, just like several important results in capital theory, on whether or not agents are altruistically linked in particular ways. For example, aggregate precautionary savings is not a necessary property of an equilibrium in at least some overlapping generations models where agents are risk averse as was mentioned in section 5.

An open question for future work concerns when aggregate wealth accumulation increases as earnings uncertainty increases. This is a stronger property than the property proved in this paper. This question could be answered by comparing steady states or by comparing time paths of aggregate wealth accumulation. The question can be addressed in both overlapping generations as well as infinitely-lived agent models. Huggett (2000) provides a general methodology for proving such statements as well as a first result relevant for the case of independent earnings shocks. The result states properties of decision rules for wealth accumulation that guarantee that aggregate wealth accumulation increases with uncertainty. One of these properties is that the decision rule increases as uncertainty increases (see Table 1). For this reason, it would seem that a positive third derivative will be important for this question even though it may not be in itself sufficient to produce the result.

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Appendix

Proof of Lemma 1:

(1) The Theorem of the Maximum together with the strict concavity of u and V generates this result.

(2) (i) As the value function is strictly increasing and strictly concave in its first argument, the right-hand derivative exists (Rockafellar (1970, Thm. 23.1)) and is strictly decreasing in k . It is clear that $V(k, e) = V(k', e')$ when $ew + k(1 + r) = e'w + k'(1 + r)$. Thus, $V_1^+(x)$ is also strictly decreasing in e .

(2) (ii) To prove that $V_1^+(x)$ is bounded we argue as follows. First, Lemma 1(2)(i) implies that $V_1^+(x) \leq V_1^+(\underline{k}, \underline{e})$ for all $x \in X$. Second, Rockafellar (1970, Thm. 29.1) implies that any subgradient of V with respect to k is of the form $\lambda_1(1 + r)$, where λ_1 is a ‘multiplier’ associated with the resource constraint in our dynamic programming problem. Thus, we know that $V_1^+(\underline{k}, \underline{e}) \leq \lambda_1(1 + r)$ if a subgradient exists. Finally, Rockafellar (1970, Cor. 28.3.1) states Kuhn-Tucker conditions that are necessary and sufficient for a solution to concave programming problems. These conditions are stated in terms of subdifferentials so the objective function need be concave but not necessarily differentiable.

Corollary 28.3.1 applies to our dynamic programming problem as the budget set has nonempty interior. Since a solution to the dynamic programming problem exists, Corollary 28.3.1 states that there exist finite, positive ‘multipliers’ $(\lambda_1, \lambda_2, \lambda_3)$ satisfying the restriction below at the optimal levels of (c, k') . Here $h(c, k') \equiv u(c) + \beta E[V(k', e')]$, ∂h denotes the subdifferential and $(\lambda_1, \lambda_2, \lambda_3)$ are multipliers associated with the resource constraint, the liquidity constraint and the non-negativity constraint on consumption.

$$(0, 0) \in \partial h(c, k') + \lambda_1(-1, -1) + \lambda_2(0, 1) + \lambda_3(1, 0)$$

(2) (iii) First show $V_1^+(x) \geq \max\{u'(c(x))(1 + r), \beta(1 + r)E[V_1^+(k(x), e')]\}$. Clearly for $\epsilon > 0$, $V(k + \epsilon, e) - V(k, e) \geq u(c(k, e) + \epsilon(1 + r)) - u(c(k, e))$ and $V(k + \epsilon, e) - V(k, e) \geq \beta E[V(k(k, e) + \epsilon(1 + r), e')] - \beta E[V(k(k, e), e')]$ Dividing by ϵ and taking limits produces $V_1^+(x) \geq u'(c(x))(1 + r)$ and $V_1^+(x) \geq \beta(1 + r)E[V_1^+(k(x), e')]$. The second inequality relies on differentiating under the expectation operator. Billingsley (1986, Thm. 16.8) shows that $V_1^+(x)$ bounded is sufficient for this to be valid. Boundedness was established in Lemma 1(2)(ii).

Second show $V_1^+(x) \leq \max\{u'(c(x))(1 + r), \beta(1 + r)E[V_1^+(k(x), e')]\}$. The following two inequalities hold for any $\epsilon > 0$:

$$V(k + \epsilon, e) - V(k, e) \leq u'(c(k, e))\Delta c + \beta E[V_1^+(k(k, e), e')]\Delta k'$$

$$V(k + \epsilon, e) - V(k, e) \leq \max\{u'(c(k, e)), \beta E[V_1^+(k(k, e), e')]\}\epsilon(1 + r)$$

The first inequality above follows from the concavity of u and V , after defining $\Delta c \equiv c(k + \epsilon, e) - c(k, e)$ and $\Delta k' \equiv k(k + \epsilon, e) - k(k, e)$. The second inequality follows from the first as the budget constraint implies that $\Delta c + \Delta k' = \epsilon(1 + r)$. The result follows from dividing each side of the second inequality by ϵ and taking limits.

(3) Lemma 1 (2)(i) and the fact that there is idiosyncratic uncertainty implies that there is $\hat{e} \in E$ and a set $A = E \cap [0, \hat{e}]$ such that $0 < P(A) < 1$ and such that the strict inequality below holds for all $e \in A$. The two weak inequalities below follow because $\beta(1 + r) \leq 1$ and because $V_1^+(x)$ is decreasing in k .

$$V_1^+(\underline{k}, e) > E[V_1^+(\underline{k}, e')] \geq \beta(1 + r)E[V_1^+(\underline{k}, e')] \geq \beta(1 + r)E[V_1^+(k(\underline{k}, e), e')]$$

Lemma 1 (2)(iii) then implies that the following equation holds for all $e \in A$.

$$V_1^+(\underline{k}, e) = u'(c(\underline{k}, e))(1 + r) > \beta(1 + r)E[V_1^+(k(\underline{k}, e), e')]$$

The continuity of $u'(c(x))$ then implies that the result holds for $x \in [\underline{k}, a) \times \hat{e}$ for some $a > \underline{k}$. As $V_1^+(x)$ is decreasing in e and as asset holding is at the borrowing constraint, the result must hold for all $x \in B = [\underline{k}, a) \times A$.

(4) We first establish that $k(k, e) < k_1$ on a set of positive probability when $k = k_1$. Suppose by way of contradiction that this does not hold. Thus, $k(k_1, e) \geq k_1$ for all $e \in A$ where $P(A) = 1$. By Lemma 1 (2) and by the fact that asset holdings are off the corner the equality below holds. The inequality holds as $\beta(1 + r) \leq 1$ and as $V_1^+(x)$ is decreasing in k .

$$V_1^+(k_1, e) = \beta(1 + r)E[V_1^+(k(k_1, e), e')] \leq E[V_1^+(k_1, e')]$$

Given that there is idiosyncratic uncertainty and that $V_1^+(x)$ is strictly decreasing in e , this cannot hold for all $e \in A$. Contradiction.

Define $\epsilon_1 = k_1 - k(k_1, e)$ using any $e \in E$ for which $\epsilon_1 > 0$ and for which any open neighborhood of e lying in E has positive probability. Set $\epsilon = \epsilon_1/2$. Since $k(x)$ is continuous (Lemma 1 (1)), there is a sufficiently small neighborhood (constructed using the Euclidean metric) around (k_1, e) lying in X such that $k(x) \leq k_1 - \epsilon$ for all x in that neighborhood. Clearly, this also holds for a neighborhood $[k_1, k_1 + \delta) \times A$ lying inside the previously constructed neighborhood. The set A can be chosen to have positive probability.

Proof of Theorem 1:

It will first be shown that in steady-state equilibrium $\beta(1+r) \leq 1$ for both part (i) and (ii) of Theorem 1. This follows from the three equations below:

$$V_1^+(x) \geq \beta(1+r)E[V_1^+(k(x), e')] \equiv \beta(1+r)E[V_1^+(x')|x]$$

$$\int_X V_1^+(x)d\lambda \geq \beta(1+r) \int_X E[V_1^+(x')|x]d\lambda$$

$$1 \geq \beta(1+r)$$

The first equation above holds by Lemma 1 (2) and by taking expectations with respect to x' . The second equation integrates the first using the steady-state distribution λ . By Lemma 1 (2) these integrals are finite. The third equation follows from the second after noting two facts. First, Stokey and Lucas (1989, Thm. 8.3) implies that $\int_X E[V_1^+(x')|x]d\lambda = \int_X V_1^+(x')d\lambda^*$, where $\lambda^*(B) \equiv \int_X P(x, B)d\lambda$ for $B \in \mathcal{X}$. Second, the definition of steady-state equilibrium implies that $\lambda^* = \lambda$.

We now show that when there is idiosyncratic uncertainty $\beta(1+r) \neq 1$. Suppose by way of contradiction that $\beta(1+r) = 1$. From this we will show that there is a set B of positive measure ($\lambda(B) > 0$) on which the borrowing constraint binds. The contradiction then follows from the four inequalities below:

$$\int_B V_1^+(x)d\lambda > \beta(1+r) \int_B E[V_1^+(x')|x]d\lambda$$

$$\int_{X/B} V_1^+(x)d\lambda \geq \beta(1+r) \int_{X/B} E[V_1^+(x')|x]d\lambda$$

$$\int_X V_1^+(x)d\lambda > \beta(1+r) \int_X E[V_1^+(x')|x]d\lambda$$

$$\beta(1+r) < 1$$

We now show that $\lambda(B) > 0$. From Lemma 1 (3) there is a set $B = [\underline{k}, a) \times A \in \mathcal{X}$ such that $x \in B$ implies that $V_1^+(x) > \beta(1+r)E[V_1^+(x')|x]$. Define $k_1 \equiv \sup\{k : \lambda([k, \infty) \times E) = 1\}$. Note that $\lambda([k_1, a) \times E) > 0$ for any $a > k_1$ by the definition of k_1 and that $\lambda([k_1, a) \times A) = P(A)\lambda([k_1, a) \times E)$ by the independence of shocks. We now prove by way of contradiction that $k_1 = \underline{k}$. Define the set $D = [\underline{k}, k_1) \times E$. Lemma 1 (4) implies that $P(x, D) = 1$ for all $x \in [k_1, a) \times A$, where $P(A) > 0$. Since $P(x, D) = 1$ on a set of positive measure (i.e. $\lambda([k_1, a) \times A) > 0$), we have that $\lambda(D) = \int_X P(x, D)d\lambda > 0$. This proves that $k_1 = \underline{k}$ and that $\lambda([\underline{k}, a) \times E) > 0$ for any $a > \underline{k}$. As shocks are independent $\lambda(B) \equiv \lambda([\underline{k}, a) \times A) = P(A)\lambda([\underline{k}, a) \times E) > 0$.

It will now be shown that when there is no idiosyncratic uncertainty $\beta(1+r) \geq 1$. Suppose that $\beta(1+r) < 1$. Define B as the set of $x \in X$ such that $V_1^+(x) > \beta(1+r)V_1^+(k(x), 1)$. Define X/B as the set of $x \in X$ such that $V_1^+(x) = \beta(1+r)V_1^+(k(x), 1)$. As $V_1(x)$ is decreasing in its first argument, $k(x) < k$ within X/B except on a set of measure zero. Within the set B , $k(x) = \underline{k}$ and thus $k(x) \leq k$. These two results and the fact that $\underline{k} < K$ imply that the inequality below holds. The equality is, once again, due to Stokey and Lucas (1989, Thm. 8.3) and to the fact that λ is time invariant in steady state. This establishes the contradiction.

$$\int_X k d\lambda > \int_X k(x) d\lambda \equiv \int_X E[k'|x] d\lambda = \int_X k' d\lambda$$

Proof of Theorem 2:

We first prove the necessity of a binding liquidity constraint for aggregate precautionary savings by contraposition. Thus, suppose that the constraint does not bind for a positive mass of agents in a steady-state equilibrium. Then the first equation below holds except on a set of measure zero. The second equation below follows by integration as the value of the integral does not depend on the value of the function on a set of measure zero. Lemma 1 (2) guarantees that these integrals are finite. The third equation follows from two facts. First, apply Stokey and Lucas (1989, Thm. 8.3) to get that $\int_X E[V_1^+(x')|x] d\lambda = \int_X V_1^+(x) d\lambda^*$, where $\lambda^*(B) \equiv \int_X P(x, B) d\lambda$. Second, note that in steady-state equilibrium $\lambda^* = \lambda$. The third equation completes the proof by contraposition as the negation of $\beta(1+r) < 1$ is $\beta(1+r) \geq 1$.

$$\begin{aligned} V_1^+(x) &= \beta(1+r)E[V_1^+(x')|x] \\ \int_X V_1^+(x) d\lambda &= \beta(1+r) \int_X E[V_1^+(x')|x] d\lambda \\ 1 &= \beta(1+r) \end{aligned}$$

We now prove the sufficiency of a binding liquidity constraint for aggregate precautionary saving. Thus, suppose that the constraint binds for a positive mass of agents. Let B be the set of states for which the constraint binds and X/B be the set for which the constraint does not bind. We then know that the first two equations hold after integrating the necessary condition from Lemma 1 (2) over the set B and its complement X/B . The third equation below follows by combining the previous two equations.

$$\int_B V_1^+(x) d\lambda > \beta(1+r) \int_B E[V_1^+(x')|x] d\lambda$$

$$\int_{X/B} V_1^+(x) d\lambda = \beta(1+r) \int_{X/B} E[V_1^+(x')|x] d\lambda$$

$$\int_X V_1^+(x) d\lambda > \beta(1+r) \int_X E[V_1^+(x')|x] d\lambda$$

The result follows from this last equation from applying, once again, Stokey and Lucas (1989, Thm. 8.3).

Table 1: Results Relevant for Overlapping Generations Models

Authors	Result	Key Assumptions	Model Periods
Leland (1968)	optimal decision rule	$u' > 0$, u' convex	$J = 2$
Sandmo (1970)	$k(x, j; \theta)$ increases in θ	u strictly concave	
Rothschild and Stiglitz (1971)			
Mirman (1971)			
Dreze and Modigliani (1972)			
Diamond and Stiglitz (1974)			
Miller (1975, 1976)	optimal decision rule	$u' > 0$, u' convex	J arbitrary
Sibley (1975)	$k(x, j; \theta)$ increases in θ	u strictly concave	
Schechtman (1976)			
Mendelson and Amihud (1982)			
Huggett (2000)	expected wealth holding profile increases in θ	$k(x, j; \theta)$ increases in x and θ and is convex in x	J arbitrary

Table 2: Results Relevant for Infinitely-Lived Agent Models

Authors	Result	Key Assumptions	Model Periods
Mendelson and Amihud (1982)	$\lim_{t \rightarrow \infty} k_t = \infty$ almost surely	u strictly concave $u' > 0$ and $r \geq \delta$	$J = \infty$
Sotomayor (1984)			
Laitner (1979)	\exists a steady state equilibrium with $r < \delta$	$u(c) = c^{(1-\sigma)}/(1-\sigma)$	$J = \infty$
Clarida (1990)	\exists a steady state equilibrium with $r < \delta$	$u' > 0$, $u'' < 0$ and $u''c/u'$ is asymptotically bounded	$J = \infty$
Huggett and Ospina	steady state capital stock is larger with uncertainty than without	u strictly concave and $u' > 0$	$J = \infty$
Huggett and Ospina	steady state capital stock is larger with uncertainty than without iff the liquidity constraint binds for some agents	u strictly concave and $u' > 0$	$J = \infty$