Life-Cycle Model

Motivation: We now combine the production structure of the Solow growth model with optimizing consumers. This will give us some of the same insights of growth theory with the ability to conduct policy analysis as it is done in microeconomics.

Model Ingredients:

1. Utility: $U(c_{yt}, c_{ot+1}) = \alpha \log c_{yt} + (1-\alpha) \log c_{ot+1}$

2. Demographic Structure: agents live two periods and die. They produce children that live for two periods and die. There are as many young agents as old agents each time period.

2. Endowments: each young agent has one unit of work time, whereas each old agent cannot work. At

the beginning of time there are some initial old agents who own all the physical capital .

4. Technology:

$$C_t + I_t = Y_t = F(K_t, L_t) = AK_t^{\beta} L_t^{1-\beta}$$

$$K_{t+1} = K_t(1 - \delta) + I_t$$

Intuition:

Agents enjoy consuming both when young and when old. Given that they have no labor income when old, how can they consume in old age? This can only come from retirement saving.

The model does not have a government running a social security system. The model does not have altruistic links between parents and children.

How does K_t and k_t evolve?

$c_{yt} = \alpha w_t$ and $c_{0t+1} = \frac{(1-\alpha)w_t}{1/(1+r_{t+1})}$

$$a_{t+1} = w_t - c_t = (1 - \alpha)w_t$$

$$K_{t+1} = Na_{t+1} = N(1-\alpha)w_t = N(1-\alpha)(1-\beta)AK_t^{\beta}N^{-\beta}$$

$$k_{t+1} = (1 - \alpha)(1 - \beta)Ak_t^{\beta}$$

Summary:

We just worked out that the savings of young agents determines how the capital-labor ratio evolves over time. Once we know this, we can easily determine how many other variables evolve over time. The logic follows closely the logic for how output, wages and interest rates evolve in the Solow model.

How do Wages Move?

Capital K_t and labor $L_t = N$ are known. Inputs are supplied inelastically at this date.

Supply curve of labor is inelastic, whereas demand is marginal product of labor.

Implication: wage is where supply and demand cross

$$w_t = MPL = (1 - \beta) A k_t^{\beta}$$

Summary:

1. Capital-Labor Ratio: $k_{t+1} = (1-\alpha)(1-\beta)Ak_t^{\beta}$

2. Output-Labor Ratio: $y_t = F(k_t, 1) = Ak_t^{\beta}$

3. Wage: $w_t = MPL = (1-\beta)Ak_t^{\beta}$

4. Investment: $i_t = k_{t+1} - k_t(1-\delta)$

Investment Paths are Tricky:

$$i_t = k_{t+1} - k_t(1-\delta) = [k_{t+1} - k_t] + \delta k_t$$

Graphically: Gap in law of motion + depreciated capital term

Upshot: It is possible that over time i_t may at first increase and then decrease to a steady state level!

Main Properties:

1. One steady state with a positive capital-labor ratio.

2. Convergence to this steady state.

3. Factor prices are marginal products

A One-Time Immigration:

Israel in late 1980's experienced a large immigration of Soviet Jews. There was a relatively short window for this immigration. We will view this as a one-time, permanent increase in the population from within the Life-Cycle model.

Implications from Life-Cycle model??

Welfare Analysis:

In economics it is standard to use the Pareto criterion. We say that a feasible allocation is Pareto efficient if there is no other feasible allocation that makes some agent strictly better off and makes no other agent worse off. We will ask whether or not equilibria within the Life-Cycle model are Pareto efficient.

PROPOSITION: Consider the life-cycle model. Assume that agents born at any time $t \geq 1$ have any $u(c_{yt}, c_{ot+1})$ that is increasing and has a well-defined marginal rate of substitution. Assume that $F_t(K_t, L_t)$ is allowed to change deterministically over time but that F_t and δ imply that the capital stock and output must remain bounded. If the allocation produced by the model has $1 + r_t > 1 + \epsilon$ for all time periods $t \geq 1$ for some number $\epsilon > 0$, then the allocation produced by any such model is Pareto efficient.

Step 1: Argue any Pareto improvement is infeasible.

Step 2: Try to improve utility for the old. Give $\Delta > 0$

Step 3: Compensation to young:

$$\Delta \times MRS(c_{yt}, c_{ot+1}) = \Delta \times (1 + r_{t+1})$$

Step 4: (Snowball Effect) $\Delta \times (1 + r_{t+1}) \times MRS(c_{yt+1}, c_{ot+2})$

Conclusion:

The initial “gift” of $\Delta > 0$ to some generation of old agents is not feasible. The young need to be compensated. This compensation grows with each generation. It eventually becomes larger than GDP! Thus, we conclude this potential Pareto improvement leads to a violation of resource feasibility!

Conclusion Continued:

Our analysis of the Life-Cycle model concludes that, at least under the conditions of the Proposition, the competitive market structure leads to an allocation that cannot be improved according to the Pareto criteria. If we later use this model to analyze policy choices, then we will end up concluding that there is no clear-cut role for government policy to improve the life of the people who live in the model.

Connection to Golden Rule:

Interest Rate Condition: $1 + r_t > 1$

Simple Model: If there is no technological change then being below the Golden Rule is equivalent to a positive real interest rate each period!