

Supplement to “Communication and Learning:” Omitted Proofs

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Proof of Lemma A.2: Let $\hat{U}_t(m_t, \omega)$ be the same type of continuation payoff as $U_t(m_{t-1}, \omega)$ defined in (A.3), but this time assuming that t sends message m_t . Clearly, $\hat{U}_t(m_t, \omega)$ and $U_t(m_{t-1}, \omega)$ are related so that

$$U_t(m_{t-1}, \omega) = \Pr(s_t = 0|\omega)\hat{U}_t(\sigma_t(m_{t-1}, 0), \omega) + \Pr(s_t = 1|\omega)\hat{U}_t(\sigma_t(m_{t-1}, 1), \omega)$$

Also, given two messages m_{t-1} and m'_{t-1} with $x_t(m_{t-1}) > x_t(m'_{t-1})$ and any $\tau \geq 0$, we define the set $Z^{t, t+\tau}(m_{t-1}, m'_{t-1})$ in the following way

$$Z^{t, t+\tau}(m_{t-1}, m'_{t-1}) = \{s^{t, t+\tau} \in \{0, 1\}^{\tau+1} \text{ such that} \\ a_{t+\tau'}(m_{t-1}, s^{t, t+\tau'-1}) > a_{t+\tau'}(m'_{t-1}, s^{t, t+\tau'-1}) \text{ for every } \tau' = 1, \dots, \tau + 1\} \quad (\text{S.1})$$

We are now ready to proceed with the argument, which for convenience is divided into 3 separate steps.

Step 1: Fix an M and σ^* as in Remark A.1. Let $m_{\bar{t}-1}$ and $m'_{\bar{t}-1}$ be two messages such that $x_{\bar{t}}(m_{\bar{t}-1}) > x_{\bar{t}}(m'_{\bar{t}-1})$. Then, $\forall \omega \in \{0, 1\}$, it must be that

$$U_{\bar{t}}(m_{\bar{t}-1}, \omega) - U_{\bar{t}}(m'_{\bar{t}-1}, \omega) = \\ (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \sum_{s^{\bar{t}, \bar{t}+t-1} \in Z^{\bar{t}, \bar{t}+t-1}(m_{\bar{t}-1}, m'_{\bar{t}-1})} \Pr(s^{\bar{t}, \bar{t}+t-1}|\omega) \\ [(\omega - a_{\bar{t}+t}(m'_{\bar{t}-1}, s^{\bar{t}, \bar{t}+t-1}))^2 - (\omega - a_{\bar{t}+t}(m_{\bar{t}-1}, s^{\bar{t}, \bar{t}+t-1}))^2]$$

To keep notation usage down, during the proof of this step we let $m_{\bar{t}-1} = m$ and $m'_{\bar{t}-1} = m'$.

Suppose that for some $t \geq 0$ there is a sequence $s^{\bar{t}, \bar{t}+t} = (s_{\bar{t}}, \dots, s_{\bar{t}+t})$ such that: (a) $a_{\bar{t}+\tau}(m, s_{\bar{t}}, \dots, s_{\bar{t}+\tau-1}) > a_{\bar{t}+\tau}(m', s_{\bar{t}}, \dots, s_{\bar{t}+\tau-1})$ for every $\tau \leq t$, and (b) $a_{\bar{t}+t+1}(m, s_{\bar{t}}, \dots, s_{\bar{t}+t}) \leq a_{\bar{t}+t+1}(m', s_{\bar{t}}, \dots, s_{\bar{t}+t})$. (If such a sequence cannot be found then there is nothing to prove.) We need to consider two cases.

(i) After the sequence of signals $s^{\bar{t}, \bar{t}+t} = (s_{\bar{t}}, \dots, s_{\bar{t}+t})$, player $\bar{t} + t$ sends the same message (say \tilde{m}) both when player \bar{t} behaves according to m and when player \bar{t} behaves according to m' . Clearly, after the sequence $s^{\bar{t}, \bar{t}+t}$, the messages m and m' will induce the same action in every period $\bar{t} + t + \tau$ for any $\tau \geq 1$. Hence in this case there is nothing further to prove.

(ii) After the sequence of signals $s^{\bar{t}, \bar{t}+t}$, player $\bar{t} + t$ sends message \tilde{m} if player \bar{t} behaves according to m and message $\tilde{m}' \neq \tilde{m}$ if player \bar{t} behaves according to m' . Also, let \tilde{y} (resp. \tilde{y}') denote the beginning-of-period belief of player $\bar{t} + t + 1$ when he receives \tilde{m} (resp. \tilde{m}'). Of course, $\tilde{y}' \geq \tilde{y}$ since

$$\tilde{y} - \alpha = a_{\bar{t}+t+1}(m, s^{\bar{t}, \bar{t}+t}) \leq a_{\bar{t}+t+1}(m', s^{\bar{t}, \bar{t}+t}) = \tilde{y}' - \alpha$$

Let \tilde{x} denote the end-of-period belief of player $\bar{t} + t$ when player \bar{t} behaves according to m and the sequence of realized signals is $s^{\bar{t}, \bar{t}+t}$. Also let \tilde{x}' denote the end-of-period belief of player $\bar{t} + t$ when player \bar{t} behaves according to m' and the sequence of realized signals is $s^{\bar{t}, \bar{t}+t}$. Notice that $\tilde{x} > \tilde{x}'$ since by assumption $a_{\bar{t}+t}(m, s_{\bar{t}+1}, \dots, s_{\bar{t}+t-1}) > a_{\bar{t}+t}(m', s_{\bar{t}+1}, \dots, s_{\bar{t}+t-1})$.

Let $\mathcal{T}_{\bar{t}+t} \subset M_{\bar{t}+t-1} \times \{0, 1\}$ denote the set of types (a type being a message received-signal observed pair) of player $\bar{t} + t$ who send message \tilde{m} . Also, let $\tilde{X}_{\bar{t}+t}^E$ denote the set of corresponding beliefs, so that $\tilde{X}_{\bar{t}+t}^E = \bigcup_{(m_{\bar{t}+t-1}, s_{\bar{t}+t}) \in \mathcal{T}_{\bar{t}+t}} x_{\bar{t}+t}^E(m_{\bar{t}+t-1}, s_{\bar{t}+t})$. Define $\mathcal{T}'_{\bar{t}+t}$ and $\tilde{X}'_{\bar{t}+t}$ in a similar way (replace \tilde{m} with \tilde{m}').

Since each player uses Bayes' rule to compute his beginning-of-period beliefs, \tilde{y} belongs to the convex hull of $\tilde{X}_{\bar{t}+t}^E$ and \tilde{y}' belongs to the convex hull of $\tilde{X}_{\bar{t}+t}^{E'}$. Notice that $\tilde{x} \in \tilde{X}_{\bar{t}+t}^E$ and $\tilde{x}' \in \tilde{X}_{\bar{t}+t}^{E'}$, and recall that $\tilde{x} > \tilde{x}'$. This, together with $\tilde{y}' \geq \tilde{y}$, imply that one of the following two cases, (a) and (b), must be true.

(a) We can find three types $(m_{\bar{t}+t-1}^1, s_{\bar{t}+t}^1)$, $(m_{\bar{t}+t-1}^2, s_{\bar{t}+t}^2)$ and $(m_{\bar{t}+t-1}^3, s_{\bar{t}+t}^3)$ such that

$$x^{(1)} = x_{\bar{t}+t}^E(m_{\bar{t}+t-1}^1, s_{\bar{t}+t}^1) < x^{(2)} = x_{\bar{t}+t}^E(m_{\bar{t}+t-1}^2, s_{\bar{t}+t}^2) < x^{(3)} = x_{\bar{t}+t}^E(m_{\bar{t}+t-1}^3, s_{\bar{t}+t}^3) \quad (\text{S.2})$$

and, the two extreme types $(m_{\bar{t}+t-1}^1, s_{\bar{t}+t}^1)$ and $(m_{\bar{t}+t-1}^3, s_{\bar{t}+t}^3)$ send the same message, equal to either \tilde{m} or \tilde{m}' , while the intermediate type sends the other. Suppose that the extreme types send \tilde{m} and $(m_{\bar{t}+t-1}^2, s_{\bar{t}+t}^2)$ sends \tilde{m}' (mutatis mutandis, the reverse case is identical). Since in equilibrium no type can have a profitable deviation when selecting which message to send, it must be that for $k = 1, 3$

$$x^{(k)}\hat{U}_{\bar{t}+t}(\tilde{m}, 1) + (1 - x^{(k)})\hat{U}_{\bar{t}+t}(\tilde{m}, 0) \geq x^{(k)}\hat{U}_{\bar{t}+t}(\tilde{m}', 1) + (1 - x^{(k)})\hat{U}_{\bar{t}+t}(\tilde{m}', 0) \quad (\text{S.3})$$

and

$$x^{(2)}\hat{U}_{\bar{t}+t}(\tilde{m}, 1) + (1 - x^{(2)})\hat{U}_{\bar{t}+t}(\tilde{m}, 0) \leq x^{(2)}\hat{U}_{\bar{t}+t}(\tilde{m}', 1) + (1 - x^{(2)})\hat{U}_{\bar{t}+t}(\tilde{m}', 0) \quad (\text{S.4})$$

Recall that by (S.2) we have $x^{(1)} < x^{(2)} < x^{(3)}$. Thus inequalities (S.3) and (S.4) can only be satisfied if $\hat{U}_{\bar{t}+t}(\tilde{m}, \omega) = \hat{U}_{\bar{t}+t}(\tilde{m}', \omega)$, $\forall \omega \in \{0, 1\}$. Therefore, after the sequence $s^{\bar{t}, \bar{t}+t}$ player \bar{t} receives the same expected continuation payoff regardless of ω and regardless of whether he behaves according to m or behaves according to m' . This concludes the argument in case (a).

(b) There are four types, $(m_{\bar{t}+t-1}^1, s_{\bar{t}+t}^1)$, $(m_{\bar{t}+t-1}^2, s_{\bar{t}+t}^2)$, $(m_{\bar{t}+t-1}^3, s_{\bar{t}+t}^3)$ and $(m_{\bar{t}+t-1}^4, s_{\bar{t}+t}^4)$ such that

$$x_{\bar{t}+t}^E(m_{\bar{t}+t-1}^1, s_{\bar{t}+t}^1) = x_{\bar{t}+t}^E(m_{\bar{t}+t-1}^2, s_{\bar{t}+t}^2) = \tilde{x} \quad \text{and} \quad x_{\bar{t}+t}^E(m_{\bar{t}+t-1}^3, s_{\bar{t}+t}^3) = x_{\bar{t}+t}^E(m_{\bar{t}+t-1}^4, s_{\bar{t}+t}^4) = \tilde{x}' \quad (\text{S.5})$$

and the two types $(m_{\bar{t}+t-1}^1, s_{\bar{t}+t}^1)$ and $(m_{\bar{t}+t-1}^3, s_{\bar{t}+t}^3)$ send message \tilde{m} while the two types $(m_{\bar{t}+t-1}^2, s_{\bar{t}+t}^2)$ and $(m_{\bar{t}+t-1}^4, s_{\bar{t}+t}^4)$ send message \tilde{m}' .

Again, in equilibrium no type can have a profitable deviation when selecting which message to send. Recalling that $\tilde{x} > \tilde{x}'$, it is then immediate to see that this implies that $\hat{U}_{\bar{t}+t}(\tilde{m}, \omega) = \hat{U}_{\bar{t}+t}(\tilde{m}', \omega)$, $\forall \omega \in \{0, 1\}$. Therefore, after the sequence $s^{\bar{t}, \bar{t}+t}$ player \bar{t} receives the same expected continuation payoff regardless of ω and regardless of whether he behaves according to m or behaves according to m' . This closes the argument in case (b) and hence concludes the proof of Step 1.

Step 2: Fix an M and σ^* as in Remark A.1. For any $\eta > 0$ there exists an $\varepsilon > 0$ such that the following is true for every t . Suppose that m_{t-1} and m'_{t-1} are two messages in M_{t-1} with $\min\{x_t(m_{t-1}), x_t(m'_{t-1})\} > 1 - \varepsilon$ and $x_t(m_{t-1}) \neq x_t(m'_{t-1})$. Then

$$|U_t(m_{t-1}, 1) - U_t(m'_{t-1}, 1)| < \eta$$

Fix η and choose $\varepsilon > 0$ such that $\varepsilon(1 - \alpha)^2/(1 - \varepsilon) < \eta$. By hypothesis, we can find two messages m_{t-1} and m'_{t-1} in M_{t-1} with $x_t(m_{t-1}) > 1 - \varepsilon$ and $x_t(m'_{t-1}) > 1 - \varepsilon$. To keep notation usage down, during the proof of this step we let $m_{t-1} = m$, $m'_{t-1} = m'$, $x_t(m_{t-1}) = x$ and $x_t(m'_{t-1}) = x'$. Without loss of generality, assume $x > x'$.

Since player t must have no incentives to deviate from his equilibrium strategy after observing s_t , taking averages, the following two inequalities must be satisfied

$$x U_t(m, 1) + (1 - x)U_t(m, 0) \geq x U_t(m', 1) + (1 - x)U_t(m', 0)$$

and

$$x'U_t(m, 1) + (1 - x')U_t(m, 0) \leq x'U_t(m', 1) + (1 - x')U_t(m', 0)$$

Therefore, for some $\bar{x} \in [x', x]$ it must be that

$$\bar{x}U_t(m, 1) + (1 - \bar{x})U_t(m, 0) = \bar{x}U_t(m', 1) + (1 - \bar{x})U_t(m', 0)$$

Hence, using the fact that $\bar{x} > 1 - \varepsilon$, we conclude that

$$|U_t(m, 1) - U_t(m', 1)| = \frac{(1 - \bar{x})}{\bar{x}} |U_t(m', 0) - U_t(m, 0)| < \frac{\varepsilon}{1 - \varepsilon} |U_t(m', 0) - U_t(m, 0)| \quad (\text{S.6})$$

Notice now that, using (3), it is trivial that no player will ever choose an action outside $[-\alpha, 1 - \alpha]$. From (A.3) it then follows directly that the continuation payoff $U_t(\cdot, 0)$ is bounded above by 0 and below by $-(1 - \alpha)^2$. It is then obvious that $|U_t(m', 0) - U_t(m, 0)| < (1 - \alpha)^2$. Hence, because of the way ε was chosen, (S.6) is enough to prove the claim in Step 2.

Step 3: Fix an M and σ^* as in Remark A.1. For any $\eta > 0$ there exists an $\varepsilon > 0$ such that the following is true for every t . Suppose that m_{t-1} and m'_{t-1} are two messages in M_{t-1} with $\min\{x_t(m_{t-1}), x_t(m'_{t-1})\} > 1 - \varepsilon$ and $x_t(m_{t-1}) \neq x_t(m'_{t-1})$. Then

$$|U_t(m_{t-1}, 0) - U_t(m'_{t-1}, 0)| < \eta$$

We proceed by contradiction. Then by hypothesis there must exist an η such that for every ε we can find messages m_{t-1} and m'_{t-1} (for some t) such that $\min\{x_t(m_{t-1}), x_t(m'_{t-1})\} > 1 - \varepsilon$, $x_t(m_{t-1}) \neq x_t(m'_{t-1})$ and

$$|U_t(m_{t-1}, 0) - U_t(m'_{t-1}, 0)| > \eta \quad (\text{S.7})$$

To keep notation usage down, during the proof of this step we let $m_{t-1} = m$, $m'_{t-1} = m'$, $x_t(m_{t-1}) = x$ and $x_t(m'_{t-1}) = x'$. Without loss of generality, assume $x > x'$. From Step 1 we know that

$$\begin{aligned} U_t(m, 0) - U_t(m', 0) = & \\ (1 - \delta) \sum_{\tau=1}^{\infty} \delta^{\tau-1} & \sum_{s^{t, t+\tau-1} \in Z^{t, t+\tau-1}(m, m')} \Pr(s^{t, t+\tau-1} | \omega = 0) \\ & [a_{t+\tau}(m', s^{t, t+\tau-1})^2 - a_{t+\tau}(m, s^{t, t+\tau-1})^2] \end{aligned} \quad (\text{S.8})$$

Just as in the proof of Step 2, the difference between continuation payoffs (appropriately normalized) conditional on $\omega = 0$ is bounded above by $(1 - \alpha)^2$. Therefore, from (S.8) we conclude that for any $T \geq 1$ it must be that

$$\begin{aligned} |U_t(m, 0) - U_t(m', 0)| \leq & \\ (1 - \delta) \sum_{\tau=1}^T \delta^{\tau-1} & \sum_{s^{t, t+\tau-1} \in Z^{t, t+\tau-1}(m, m')} \Pr(s^{t, t+\tau-1} | \omega = 0) \\ & |a_{t+\tau}(m', s^{t, t+\tau-1})^2 - a_{t+\tau}(m, s^{t, t+\tau-1})^2| + \delta^T (1 - \alpha)^2 \end{aligned} \quad (\text{S.9})$$

Since T can always be chosen so that $\delta^T(1-\alpha)^2 < \eta/2$, we conclude that there exists a T such that

$$|U_t(m, 0) - U_t(m', 0)| < (1-\delta) \sum_{\tau=1}^T \delta^{\tau-1} \sum_{s^{t,t+\tau-1} \in Z^{t,t+\tau-1}(m,m')} \Pr(s^{t,t+\tau-1} | \omega = 0) |a_{t+\tau}(m', s^{t,t+\tau-1})^2 - a_{t+\tau}(m, s^{t,t+\tau-1})^2| + \frac{\eta}{2} \quad (\text{S.10})$$

Inequalities (S.7) and (S.10) directly imply that

$$(1-\delta) \sum_{\tau=1}^T \delta^{\tau-1} \sum_{s^{t,t+\tau-1} \in Z^{t,t+\tau-1}(m,m')} \Pr(s^{t,t+\tau-1} | \omega = 0) |a_{t+\tau}(m', s^{t,t+\tau-1})^2 - a_{t+\tau}(m, s^{t,t+\tau-1})^2| > \frac{\eta}{2} \quad (\text{S.11})$$

However, inequality (S.11) implies that for some $\bar{\tau} = 1, \dots, T$ and for some sequence of signals $\bar{s}^{t,t+\bar{\tau}-1} \in Z^{t,t+\bar{\tau}-1}(m, m')$ it must be the case that

$$|a_{t+\bar{\tau}}(m', \bar{s}^{t,t+\bar{\tau}-1})^2 - a_{t+\bar{\tau}}(m, \bar{s}^{t,t+\bar{\tau}-1})^2| > \frac{\eta}{2(1-\delta^T)} \quad (\text{S.12})$$

Since $\bar{s}^{t,t+\bar{\tau}-1} \in Z^{t,t+\bar{\tau}-1}(m, m')$, using (S.1) we know that $a_{t+\bar{\tau}}(m, \bar{s}^{t,t+\bar{\tau}-1}) > a_{t+\bar{\tau}}(m', \bar{s}^{t,t+\bar{\tau}-1})$. Therefore, from (S.12) we can conclude that

$$a_{t+\bar{\tau}}(m, \bar{s}^{t,t+\bar{\tau}-1}) - a_{t+\bar{\tau}}(m', \bar{s}^{t,t+\bar{\tau}-1}) > \frac{\eta}{2(1-\delta^T)} \frac{1}{|a_{t+\bar{\tau}}(m, \bar{s}^{t,t+\bar{\tau}-1}) + a_{t+\bar{\tau}}(m', \bar{s}^{t,t+\bar{\tau}-1})|} > \frac{\eta}{2(1-\delta^T)} \frac{1}{2(1-\alpha)} = \frac{\eta}{4(1-\delta^T)(1-\alpha)} \quad (\text{S.13})$$

According to (2), conditional on $\omega = 1$, the (undiscounted) difference in period $t + \bar{\tau}$ payoffs for player t arising from the two actions $a_{t+\bar{\tau}}(m, \bar{s}^{t,t+\bar{\tau}-1})$ and $a_{t+\bar{\tau}}(m', \bar{s}^{t,t+\bar{\tau}-1})$ is given by

$$[1 - a_{t+\bar{\tau}}(m', \bar{s}^{t,t+\bar{\tau}-1})]^2 - [1 - a_{t+\bar{\tau}}(m, \bar{s}^{t,t+\bar{\tau}-1})]^2 \quad (\text{S.14})$$

The quantity in (S.14) is positive since, as we just noted above, $a_{t+\bar{\tau}}(m, \bar{s}^{t,t+\bar{\tau}-1}) > a_{t+\bar{\tau}}(m', \bar{s}^{t,t+\bar{\tau}-1})$. An actual lower bound to (S.14) can be obtained by developing the square terms and using (S.13). This gives

$$[1 - a_{t+\bar{\tau}}(m', \bar{s}^{t,t+\bar{\tau}-1})]^2 - [1 - a_{t+\bar{\tau}}(m, \bar{s}^{t,t+\bar{\tau}-1})]^2 > \frac{2\alpha\eta}{4(1-\delta^T)(1-\alpha)} \quad (\text{S.15})$$

Consider now the difference $U_t(m, 1) - U_t(m', 1)$. Using Step 1 we can write

$$\begin{aligned}
 U_t(m, 1) - U_t(m', 1) = & \\
 (1 - \delta) \sum_{\tau=1}^{\infty} \delta^{\tau-1} & \sum_{s^{t,t+\tau-1} \in Z^{t,t+\tau-1}(m,m')} \Pr(s^{t,t+\tau-1} | \omega = 1) \\
 & [(1 - a_{t+\tau}(m', s^{t,t+\tau-1}))^2 - (1 - a_{t+\tau}(m, s^{t,t+\tau-1}))^2]
 \end{aligned} \tag{S.16}$$

Since $s^{t,t+\tau-1} \in Z^{t,t+\tau-1}(m, m')$, using (S.1), for every $\tau \geq 1$ it must be that

$$[1 - a_{t+\tau}(m', s^{t,t+\tau-1})]^2 - [1 - a_{t+\tau}(m, s^{t,t+\tau-1})]^2 > 0 \tag{S.17}$$

Using (S.15) to deal with one term of (S.16) and (S.17) to deal with the others, we conclude that

$$U_t(m, 1) - U_t(m', 1) > (1 - \delta) \delta^T (1 - p)^T \frac{\alpha \eta}{2(1 - \delta^T)(1 - \alpha)} \tag{S.18}$$

Recall now that η is given in our contradiction hypothesis, and that T only depends on η . Furthermore, m and m' are such that $\min\{x_t(m), x_t(m')\} > 1 - \varepsilon$ with ε arbitrarily small and $x_t(m) \neq x_t(m')$. Therefore (S.18) contradicts Step 2 and the proof of the claim in Step 3 is now complete.

It is now evident that the claim in (A.5) is a direct consequence of Step 2 and Step 3. Hence the proof of Lemma A.2 is now complete. ■

Proof of Lemma A.5: We say that σ is a Markov strategy profile if for every t and t' and every pair of sequences of messages $m^t \in \mathcal{M}_{\sigma}^t$ and $m^{t'} \in \mathcal{M}_{\sigma}^{t'}$, whenever $\Pr_{\sigma}(\omega = 1 | m^t, x) = \Pr_{\sigma}(\omega = 1 | m^{t'}, x)$, then

$$\sigma_{t+1}(m^t, s_{t+1}) = \sigma_{t'+1}(m^{t'}, s_{t+1}) \quad \forall s_{t+1} \in \{0, 1\} \tag{S.19}$$

(Recall that (S.19) is true by construction when $s_{t+1} = 1$; see (A.13).)

As is standard our definition of a Markov strategy profile embodies the requirement that the players' strategies must depend only on their beliefs, and not on the histories of past messages sent. Notice also that in our definition of Markov strategies we consider only sequences of messages $m^t \in \mathcal{M}_{\sigma}^t$ that have positive probability under σ . As will become clear, there is never any need for us to specify off-path behavior in the argument below.

Because we only consider pure strategies, a Markov strategy profile can be represented in a particularly simple way. In fact, using (S.19) it is sufficient to consider a function \mathbf{f} in the set

$$F = \{\mathbf{f} : \Gamma(r, p) \rightarrow \{0, 1\}\}$$

to be interpreted in the following way.

Consider a player whose beginning-of-period belief is $z \in \Gamma(r, p)$. Then the player truthfully reveals his signal if $\mathbf{f}(z) = 0$, while babbling occurs if $\mathbf{f}(z) = 1$. We adopt this convention throughout the rest of the proof of Lemma A.5. Moreover, we let \mathbf{f}^{TR} denote the truthful strategy profile, so that $\mathbf{f}^{TR}(z) = 0$ for every $z \in \Gamma(r, p)$, and we let \mathbf{f}^B denote the babbling strategy profile, so that $\mathbf{f}^B(z) = 1$ for every $z \in \Gamma(r, p)$.

Now consider the minimization problem

$$\min_{\sigma} V_0(\sigma, x) \tag{S.20}$$

We will show that there exists an $\bar{x} < 1$ such that for every $x \in [\bar{x}, 1] \cap \Gamma(r, p)$, the value of the minimized objective function in (S.20) is $V_0(\boldsymbol{\sigma}^B, x)$, which is obviously enough to prove the claim.

Recall that, by definition, $V_0(\boldsymbol{\sigma}, x)$ is defined for an arbitrary (not necessarily equilibrium) profile $\boldsymbol{\sigma}$, although it presumes that each individual chooses his optimal action given his beliefs in the action stage.

Notice also that since (S.20) is a stationary, single agent problem, there is no loss generality in restricting attention to Markov strategy profiles. In other words, abusing notation slightly and using \mathbf{f} as an admissible argument of $V_0(\cdot)$, the value of (S.20) is equivalent to the value of the problem

$$\min_{\mathbf{f} \in F} V_0(\mathbf{f}, x) \quad (\text{S.21})$$

Clearly, the babbling strategy $\mathbf{f}^B \equiv \boldsymbol{\sigma}^B$ is a Markov strategy. In the remainder of the proof, we show that it solves (S.21) when x is sufficiently large.

The next part of the argument is divided into 3 separate steps.

Step 1: Suppose that player t has a beginning-of-period belief equal to z and that player 0 has a prior of 0.

Let \mathbf{f}' denote a Markov strategy in which player t with beliefs z truthfully reveals his signal to player $t + 1$. Let \mathbf{f}'' denote a Markov strategy in which player t with beliefs z babbles. Let $d(z)$ denote the period $t + 1$ -payoff difference to player 0 between \mathbf{f}' and \mathbf{f}'' . Then $d(z)$ is given by

$$d(z) \equiv -p \left[\frac{z(1-p)}{z(1-p) + (1-z)p} - \alpha \right]^2 - (1-p) \left[\frac{zp}{zp + (1-z)(1-p)} - \alpha \right]^2 + (z - \alpha)^2 \quad (\text{S.22})$$

and, moreover, $d(0) = d(1) = 0$, and there exists $\hat{z} \in (0, 1)$, such that $d(z) \leq 0$ for every $z \in [0, \hat{z}]$ and $d(z) \geq 0$ for every $z \in [\hat{z}, 1]$.

To verify the claim, observe that if player t reveals his signal truthfully, player $t + 1$ will choose action $[z(1-p)]/[z(1-p) + (1-z)p] - \alpha$ if player t observes signal 0, and action $zp/[zp + (1-z)(1-p)] - \alpha$ if player t observes signal 1 instead. If, on the other hand, if player t babbles, then player $t + 1$ will choose action $z - \alpha$.

Since the value $d(z)$ is just the payoff difference to player 0 between truthful reporting by player t and babbling by player t , and by assumption player 0 has prior 0, (S.22) follows immediately. The other properties in the claim are easily verified by inspection of (S.22) itself.

Step 2: Consider a subset $F^* \subset F$ of the Markov profiles defined by

$$F^* = \{ \mathbf{f} \in F \mid \text{(i) } \mathbf{f}(z) = 0 \text{ for every } z \leq \hat{z}; \text{ (ii) } \mathbf{f} \text{ is weakly increasing} \} \quad (\text{S.23})$$

where \hat{z} is as in Step 1. Then, $\min_{\mathbf{f} \in F^*} V_0(\mathbf{f}, x) = \min_{\mathbf{f} \in F} V_0(\mathbf{f}, x)$ for any prior x . Consequently, there is no loss of generality in substituting F^* for F in problem (S.21).

To prove the claim, fix a prior x and an arbitrary Markov strategy profile $\mathbf{f} \in F$. It suffices to show that there exists an $\tilde{\mathbf{f}} \in F^*$ such that

$$V_0(\tilde{\mathbf{f}}, x) \leq V_0(\mathbf{f}, x) \quad (\text{S.24})$$

Suppose that $\mathbf{f} \notin F^*$, otherwise there is nothing to prove. We break the argument into two cases. We first establish (S.24) for the case where \mathbf{f} satisfies property (i) in the definition of F^* , but not property (ii). We then establish (S.24) for the case where \mathbf{f} violates property (i).

Consider the first case. Since \mathbf{f} satisfies property (i) but violates (ii), there must exist some $z' \in (\hat{z}, 1) \cap \Gamma(r, p)$ such that $\mathbf{f}(z') = 1$. Since $\Gamma(r, p)$ is finite on any subinterval in $(0, 1)$, there must be a smallest such point $z' > \hat{z}$. We therefore consider precisely this smallest z' . Thus $\mathbf{f}(z) = 0$ for every $z < z'$.

We now show that for all $z \in (z', 1) \cap \Gamma(r, p)$ it must be that

$$V_0(\mathbf{f}^B, z) \leq V_0(\mathbf{f}, z) \quad (\text{S.25})$$

Suppose by way of contradiction that $V_0(\mathbf{f}, z'') < V_0(\mathbf{f}^B, z'')$ for some $z'' > z' > \hat{z}$. Because there is discounting, this inequality is still satisfied if we compute the discounted expected payoff of player zero in the first T periods (for some T large). Hence, we know that there exists a finite T such that

$$V_0^T(\mathbf{f}, z'') < V_0^T(\mathbf{f}^B, z'') \quad (\text{S.26})$$

where $V_0^T(\mathbf{f}, z'')$ and $V_0^T(\mathbf{f}^B, z'')$ are as defined in (A.19). Notice also that (S.26) implies that $\mathbf{f}(z'') = 0$.

To show that (S.26) is false, starting from \mathbf{f} , we construct an array of strategy profiles σ^τ with $\tau = 2, \dots, T-1$, as follows. First, $\sigma_1^\tau(m^0, 0) = 0$ (i.e., player 1 reveals his signal). Then fix $t = 2, \dots, \tau-1$ and a message $m^{t-1} \in \mathcal{M}_{\sigma^\tau}^{t-1}$ (the set $\mathcal{M}_{\sigma^\tau}^{t-1}$ can be computed inductively starting from $\mathcal{M}_{\sigma^\tau}^1 = \{0, 1\}$ using (S.27) below). Then let

$$\sigma_t^\tau(m^{t-1}, 0) = \begin{cases} 0 & \text{if } \mathbf{f}[\Pr_{\sigma^\tau}(\omega = 1 \mid m^{t-1}, z'')] = 0 \\ 1 & \text{if } \mathbf{f}[\Pr_{\sigma^\tau}(\omega = 1 \mid m^{t-1}, z'')] = 1 \end{cases} \quad (\text{S.27})$$

Of course the strategy σ_t^τ also has to specify the behavior of player t when he receives an off-path message $m^{t-1} \in M^{t-1} \setminus \mathcal{M}_{\sigma^\tau}^{t-1}$. However, this off-path behavior does not affect the payoff to player zero arising from the strategy profile σ^τ . Hence, we can arbitrarily set $\sigma_t^\tau(m^{t-1}, 0) = 1$ for $m^{t-1} \in M^{t-1} \setminus \mathcal{M}_{\sigma^\tau}^{t-1}$. Finally, for every $t \geq \tau$ and for every $m^{t-1} \in M^{t-1}$ we set $\sigma_t^\tau(m^{t-1}, 0) = 1$.

Intuitively, under the strategy profile σ^τ , the first $\tau-1$ players follow the original strategy profile \mathbf{f} , while the remaining players adopt the babbling strategy.

Recall that under the Markov strategy profile \mathbf{f} babbling occurs at z' and that $z'' > z' > \hat{z}$. Thus, under \mathbf{f} , starting from a prior of z'' , the beginning-of-period belief of every player t must be weakly greater than z' . Finally, recall from Step 1 that $d(z) \geq 0$ for every $z > \hat{z}$. Hence

$$V_0^T(\mathbf{f}, z'') \geq V_0^T(\sigma^{T-1}, z'') \geq V_0^T(\sigma^{T-2}, z'') \geq \dots \geq V_0^T(\sigma^2, z'') \geq V_0^T(\mathbf{f}^B, z'') \quad (\text{S.28})$$

which contradicts inequality (S.26) and, therefore, establishes (S.25).

To complete the verification of (S.24) when \mathbf{f} satisfies property (i) but violates (ii) of (S.23), it is now enough to define the Markov strategy profile $\tilde{\mathbf{f}} \in F^*$ as follows

$$\tilde{\mathbf{f}}(z) = \begin{cases} 0 & \text{if } z < z' \\ 1 & \text{if } z \geq z' \end{cases} \quad (\text{S.29})$$

Clearly, inequality (S.25) implies inequality (S.24) for the case $x > z'$. Also, for every $x \leq z'$, it must be that $V_0(\tilde{\mathbf{f}}, x) = V_0(\mathbf{f}, x)$. This is because under the Markov profiles \mathbf{f} and $\tilde{\mathbf{f}}$, the beginning-of-period beliefs starting from a prior of x are weakly smaller than z' for every player. Moreover, the two profiles differ only if the beginning-of-period belief is strictly greater than z' and the argument is complete.

It remains to show that (S.24) holds when \mathbf{f} violates property (i). In the light of our argument in this Step so far, it is sufficient to show that given an \mathbf{f} that violates property (i), we can find an \mathbf{f}' that satisfies property (i) and gives

$$V_0(\mathbf{f}', x) \leq V_0(\mathbf{f}, x) \quad (\text{S.30})$$

To construct this \mathbf{f}' we first define the Markov strategy profile $\bar{\mathbf{f}} \in F^*$ as

$$\bar{\mathbf{f}}(z) = \begin{cases} 0 & \text{if } z \leq \hat{z} \\ 1 & \text{if } z > \hat{z} \end{cases} \quad (\text{S.31})$$

The case $x > \hat{z}$ and $\mathbf{f}(x) = 1$ is trivial since $V_0(\bar{\mathbf{f}}, x) = V_0(\mathbf{f}, x) = -(x - \alpha)^2$, and hence we can just set $\mathbf{f}' = \bar{\mathbf{f}}$. Thus, we now turn to the remaining cases, namely $x \leq \hat{z}$ and/or $\mathbf{f}(x) = 0$.

Let $Z \subset \Gamma(r, p)$ denote the set of elements $z \leq \hat{z}$ such that $\mathbf{f}(z) = 1$. Using an induction argument similar to the one we used to establish (S.28), we can show that for every $z \in Z$

$$V_0(\bar{\mathbf{f}}, z) \leq V_0(\mathbf{f}, z) \quad (\text{S.32})$$

and the details are omitted for the sake of brevity.

Next, from \mathbf{f} and $\bar{\mathbf{f}}$, construct the strategy profile $\tilde{\sigma}$ that intuitively prescribes the following behavior. Begin with \mathbf{f} and replace the continuation strategy profile (not just the current action) with $\bar{\mathbf{f}}$ whenever, starting from a prior of x , we run into a beginning-of-period belief $z \in Z$.

Formally, proceed inductively forward as follows. First, let $\tilde{\sigma}_1(m^0, 0) = 0$ (i.e., player 1 reveals his signal). For every $t > 1$, compute the set $\mathcal{M}_{\tilde{\sigma}}^{t-1}$ inductively forward starting from $\mathcal{M}_{\tilde{\sigma}}^1 = \{0, 1\}$ and using (S.33), (S.34) and (S.35). Define $\bar{\mathcal{M}}_{\tilde{\sigma}}^{t-1} \subseteq \mathcal{M}_{\tilde{\sigma}}^{t-1}$ as follows

$$\bar{\mathcal{M}}_{\tilde{\sigma}}^{t-1} = \{m^{t-1} = (m_1, \dots, m_{t-1}) \in \mathcal{M}_{\tilde{\sigma}}^{t-1} \mid \exists \tau \leq t-1 \text{ and } z \in Z \text{ such that } \Pr_{\tilde{\sigma}}(\omega = 1 \mid (m_1, \dots, m_\tau), x) = z\} \quad (\text{S.33})$$

For every $m^{t-1} \in \bar{\mathcal{M}}_{\tilde{\sigma}}^{t-1}$ let

$$\tilde{\sigma}_t(m^{t-1}, 0) = \begin{cases} 0 & \text{if } \bar{\mathbf{f}}[\Pr_{\tilde{\sigma}}(\omega = 1 \mid m^{t-1}, x)] = 0 \\ 1 & \text{if } \bar{\mathbf{f}}[\Pr_{\tilde{\sigma}}(\omega = 1 \mid m^{t-1}, x)] = 1 \end{cases} \quad (\text{S.34})$$

and for every $m^{t-1} \in \mathcal{M}_{\tilde{\sigma}}^{t-1} \setminus \bar{\mathcal{M}}_{\tilde{\sigma}}^{t-1}$ let

$$\tilde{\sigma}_t(m^{t-1}, 0) = \begin{cases} 0 & \text{if } \mathbf{f}[\Pr_{\tilde{\sigma}}(\omega = 1 \mid m^{t-1}, x)] = 0 \\ 1 & \text{if } \mathbf{f}[\Pr_{\tilde{\sigma}}(\omega = 1 \mid m^{t-1}, x)] = 1 \end{cases} \quad (\text{S.35})$$

Of course, to conclude the description of the strategy profile $\tilde{\sigma}$ we need to describe off-path behavior. However, as before, off-path behavior does not affect the payoff to player zero of the profile $\tilde{\sigma}$. We arbitrarily set off-path behavior to be always babbling so that $\tilde{\sigma}_t(m^{t-1}, 0) = 1$ for every $m^{t-1} \in \mathcal{M}_{\tilde{\sigma}}^{t-1} \setminus \bar{\mathcal{M}}_{\tilde{\sigma}}^{t-1}$.

Given our construction of $\tilde{\sigma}$ and the fact that inequality (S.32) holds for every $z \in Z$, it follows that

$$V_0(\tilde{\sigma}, x) \leq V_0(\mathbf{f}, x) \quad (\text{S.36})$$

Observe also that by construction, $\tilde{\sigma}_t(m^{t-1}, 0) = 0$ for every t , and every $m^{t-1} \in \mathcal{M}_{\tilde{\sigma}}^{t-1}$ with

$$\Pr_{\tilde{\sigma}}(\omega = 1 \mid m^{t-1}, x) \leq \hat{z} \quad (\text{S.37})$$

The strategy profile $\tilde{\sigma}$ is not necessarily Markov. In particular, if we let z^+ denote the smallest element of $\Gamma(r, p)$ strictly greater than \hat{z} , then the following is possible. There exist $m^{t-1} \in \mathcal{M}_{\tilde{\sigma}}^{t-1}$ and $m^{t'-1} \in \mathcal{M}_{\tilde{\sigma}}^{t'-1}$

such that

$$\Pr_{\tilde{\sigma}}(\omega = 1|m^{t-1}, x) = \Pr_{\tilde{\sigma}}(\omega = 1|m^{t'-1}, x) = z^+ \quad (\text{S.38})$$

and

$$\tilde{\sigma}_t(m^{t-1}, 0) \neq \tilde{\sigma}_t(m^{t'-1}, 0) \quad (\text{S.39})$$

On the other hand, it is easy to check that if $m^{t-1} \in \mathcal{M}_{\tilde{\sigma}}^{t-1}$ and $m^{t'-1} \in \mathcal{M}_{\tilde{\sigma}}^{t'-1}$ satisfy

$$\Pr_{\tilde{\sigma}}(\omega = 1|m^{t-1}, x) = \Pr_{\tilde{\sigma}}(\omega = 1|m^{t'-1}, x) = z \quad (\text{S.40})$$

for some $z \neq z^+$, then

$$\tilde{\sigma}_t(m^{t-1}, 0) = \tilde{\sigma}_t(m^{t'-1}, 0) \quad (\text{S.41})$$

This suggests that to obtain the Markov profile \mathbf{f}' that we need, it is convenient to consider two cases. Suppose first that there exists an $\bar{m}^{t-1} \in \mathcal{M}_{\tilde{\sigma}}^{t-1}$ (for some t) such that

$$\Pr_{\tilde{\sigma}}(\omega = 1|\bar{m}^{t-1}, x) = z^+ \quad (\text{S.42})$$

and

$$\tilde{\sigma}_t(\bar{m}^{t-1}, 0) = 0 \quad (\text{S.43})$$

Let also $V_0(\tilde{\sigma}, x, \bar{m}^{t-1})$ denote the discounted continuation payoff (evaluated in period t) of player zero when the strategy profile $\tilde{\sigma}$ is adopted and the profile of messages \bar{m}^{t-1} is realized, and define the Markov strategy profile \mathbf{f}' as

$$\mathbf{f}'(z) = \begin{cases} 0 & \text{if } z \leq \hat{z} \\ 0 & \text{if } z = z^+ \text{ and } V_0(\tilde{\sigma}, x, \bar{m}^{t-1}) < -(z^+ - \alpha)^2 \\ 1 & \text{if } z = z^+ \text{ and } V_0(\tilde{\sigma}, x, \bar{m}^{t-1}) \geq -(z^+ - \alpha)^2 \\ \mathbf{f}(z) & \text{if } z > z^+ \end{cases} \quad (\text{S.44})$$

If \bar{m}^{t-1} as in (S.42) and (S.43) cannot be found, then set \mathbf{f}' as in (S.44) with $\mathbf{f}(z^+) = 1$.

By construction we have that

$$V_0(\mathbf{f}', x) \leq V_0(\tilde{\sigma}, x) \leq V_0(\mathbf{f}, x) \quad (\text{S.45})$$

Since \mathbf{f}' satisfies property (i) in (S.23), this concludes the proof of the claim in Step 2.

Step 3: *There exists a $\bar{z} < 1$ such that $V_0(\mathbf{f}^B, z) < V_0(\mathbf{f}^{TR}, z)$ for every $z \in (\bar{z}, 1)$.*

To prove the claim, fix a z , and for any T , define $\hat{V}_0^T(\mathbf{f}^{TR}, z)$ to be the payoff to player 0 with a prior of zero when players $1, \dots, T$ have the prior z and use the truthful strategies, and then players $T+1, T+2, \dots$, choose the action $1 - \alpha$. Clearly, $\hat{V}_0^T(\mathbf{f}^{TR}, z) = V_0^T(\mathbf{f}^{TR}, z) + \delta^T V_0(\mathbf{f}^B, 1)$ and can be written as

$$-(1 - \delta)(z - \alpha)^2 - (1 - \delta) \sum_{t=1}^{T-1} \delta^t \sum_{s^t \in \{0,1\}^t} \Pr(s^t | \omega = 0) [\beta_{t+1}(s^t, \sigma^{TR}, z) - \alpha]^2 - \delta^T (1 - \alpha)^2$$

Notice that $1 - \alpha$ is the worst possible action for player zero with a prior of zero. Therefore, $\hat{V}_0^T(\mathbf{f}^{TR}, z) < V_0(\mathbf{f}^{TR}, z)$.

Next, observe that

$$\hat{V}_0^T(\mathbf{f}^{TR}, 1) = V_0(\mathbf{f}^B, 1) = -(1 - \alpha)^2$$

and that using simple algebra we can write

$$\left. \frac{\partial V_0(\mathbf{f}^B, z) - \hat{V}_0^T(\mathbf{f}^{TR}, z)}{\partial z} \right|_{z=1} = -2\delta(1 - \alpha) + 2(1 - \delta)(1 - \alpha) \sum_{t=1}^{T-1} \delta^t \sum_{s^t} \frac{\Pr(s^t | \omega = 0)^2}{\Pr(s^t | \omega = 1)}$$

which is strictly positive for T large enough. Hence, there must exist a \tilde{T} and a $\tilde{z} < 1$ such that

$$V_0(\mathbf{f}^B, z) < \hat{V}_0^{\tilde{T}}(\mathbf{f}^{TR}, 1) < V_0(\mathbf{f}^{TR}, z) \quad (\text{S.46})$$

for every $z \in (\tilde{z}, 1)$. This is clearly enough to prove the claim in Step 3.

We are now ready to proceed with the final part of the proof of Lemma A.5. Fix any prior $\tilde{x} \in \Gamma(r, p)$, with $\tilde{x} > \tilde{z}$, where \tilde{z} is defined in Step 3 and let

$$V_0^*(\tilde{x}) = \inf_{\mathbf{f} \in F^*} V_0(\mathbf{f}, \tilde{x})$$

We know from Step 3 that there exists an $\eta > 0$ such that

$$V_0(\mathbf{f}^{TR}, \tilde{x}) - V_0^*(\tilde{x}) \geq V_0(\mathbf{f}^{TR}, \tilde{x}) - V_0(\mathbf{f}^B, \tilde{x}) > \eta$$

Because there is discounting and the players' payoffs are bounded, we can find a $\tilde{z} \in \Gamma(r, p)$ sufficiently large so that for every $\mathbf{f} \in F^*$ such that $\mathbf{f}(\tilde{z}) = 0$ we have that

$$V_0(\mathbf{f}, \tilde{x}) - V_0^*(\tilde{x}) > \eta/2$$

This, in turn, implies that there are only finitely many functions $\mathbf{f} \in F^*$ yielding payoff $V_0(\mathbf{f}, \tilde{x})$ within $\eta/2$ of $V_0^*(\tilde{x})$. Clearly this implies that the payoff $V_0^*(\tilde{x})$ is achieved by some strategy profile in F^* . We let $\mathbf{f}_{\tilde{x}}^*$ denote a profile (arbitrarily picked if necessary) that achieves this minimum.

Next, let \tilde{x} be the smallest value above \tilde{x} for which the strategy profile $\mathbf{f}_{\tilde{x}}^*$ prescribes babbling. Formally, define \tilde{x} as the lowest element of the (finite) set

$$\{z \in [\tilde{x}, 1] \cap \Gamma(r, p) \mid \mathbf{f}_{\tilde{x}}^*(z) = 1\}$$

Using the same argument as in the first case of the proof of the claim in Step 2 it is straightforward to show that

$$V_0(\mathbf{f}^B, x) = \min_{\mathbf{f} \in F^*} V_0(\mathbf{f}, x) \leq V_0(\boldsymbol{\sigma}, x)$$

for every $x \in [\tilde{x}, 1] \cap \Gamma(r, p)$ and every strategy profile $\boldsymbol{\sigma}$. Thus, the proof of Lemma A.5 is now complete. ■