A "Super" Folk Theorem for Dynastic Repeated Games: Technical Addendum

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T.1. A Compact Description of Strategies

Definition T.1.1. *Action-Stage Strategies: Let k be an element of I, and j be an element of I not equal to i.*

Let $(a(1),\ldots,a(\|A\|))$ be the enumeration of the elements of A of Definition A.2 and consider the indexation of the elements of X in Definition A.7, according to whether $x(\kappa)$ has $\kappa \leq \overline{\kappa}$ or not.

Recall that at the beginning of period 0 all players $\langle i \in I, 0\rangle$ receive message $m_i^0 = \emptyset$. For all players $\langle i \in I, 0\rangle$ then define

$$g_i^0(m_i^0, x^0) = \begin{cases} a_i(\ell_0) & \text{if } \; x^0 = x(\ell_0, \cdots) \\ a_i(\ell_{00}) & \text{if } \; x^0 = x(\ell_{00}, \cdot) \end{cases} \tag{T.1.1}$$

Now consider any player $\langle i, t\rangle$ with $t \geq 1$. It is convenient to distinguish between the two cases $x^t = x(\kappa)$ with $\kappa \leq \overline{\kappa}$ and with $\kappa > \overline{\kappa}$.

For any $i \in I$ and $t \geq 1$ whenever $x^t = x(\kappa)$ with $\kappa \leq \overline{\kappa}$ define[T.1]

$$g_i^t(m_i^t, x^t) = \begin{cases} a_i(\hat{\ell}) & \text{if } \; m_i^t = m^* & \text{and } \; x^t = x(\cdot, \hat{\ell}, \cdots) \\ \breve{a}_i^j(j_\ell) & \text{if } \; m_i^t = \breve{m}_j & \text{and } \; x^t = x(\cdots, j_\ell, \cdots) \\ a_i(\ell_k) & \text{if } \; m_i^t = \overline{m}^k & \text{and } \; x^t = x(\cdots, \ell_k, \cdots) \\ a_i(k_\ell) & \text{if } \; m_i^t \in \underline{M}(k,t) & \text{and } \; x^t = x(\cdots, k_\ell, \cdots) \end{cases} \tag{T.1.2}$$

For any $i \in I$, $t \geq 1$ and m_i^t, whenever $x^t = x(\kappa)$ with $\kappa > \overline{\kappa}$ define

$$g_i^t(m_i^t, x^t) = a_i(\ell^*) \; \text{ if } \; x^t = x(\cdot, \ell^*) \tag{T.1.3}$$

Definition T.1.2. *Message-Stage Strategies: Let k be any element of I, and j be any element of I not equal to i.*

We begin with period $t = 0$. Recall that $m_i^0 = \emptyset$ for all $i \in I$. Let also $g^0(m^0, x^0) = (g_1^0(m_1^0, x^0), \ldots, g_n^0(m_n^0, x^0))$, and define $g_{-k}^0(m^0, x^0)$ in the obvious way.

We let

$$\mu_i^0(m_i^0, x^0, a^0, y^0) = \begin{cases} \breve{m}^j & \text{if } \; a^0 = g^0(m^0, x^0) & \text{and } \; y^0 = y(j) \\ \underline{m}^{i,T} & \text{if } \; a^0 = g^0(m^0, x^0) & \text{and } \; y^0 = y(i) \\ \underline{m}^{k,T} & \text{if } \; a_{-k}^0 = g_{-k}^0(m^0, x^0) & \text{and } \; a_k^0 \neq g_k^0(m^0, x^0) \\ m^* & & \text{otherwise} \end{cases} \tag{T.1.4}$$

For the periods $t \geq 1$ it is convenient to distinguish between several cases. Assume first that $x^t = x^t(\kappa)$ with $\kappa > \overline{\kappa}$. Let

$$\mu_i^t(m_i^t, x^t, a^t, y^t) = \begin{cases} m_i^t & \text{if } \; x^t = x(\cdot, \ell^*) & \text{and } \; a^t = a(\ell^*) \\ \underline{m}^{k,T} & \text{if } \; x^t = x(\cdot, \ell^*), \;\; a_{-k}^t = a_{-k}(\ell^*) & \text{and } \; a_k^t \neq a_k(\ell^*) \\ m^* & & \text{otherwise} \end{cases} \tag{T.1.5}$$

[T.1] Notice that the third case in (T.1.2) can only possibly apply when $t \geq T + 1$.

Now consider the case $x^t = x^t(\kappa)$ with $\kappa \leq \overline{\kappa}$. We divide this case into several subcases, according to which message player $\langle i,t\rangle$ has received. We begin with $m_i^t = m^*$. Let[T.2]

$$\mu_i^t(m^*, x^t, a^t, y^t) = \begin{cases} \breve{m}^j & \text{if } x^t = x(\cdot, \hat{\ell}, \cdots),\ a^t = a(\hat{\ell}) & \text{and } y^t = y(j) \\ \nu(\underline{M}(i,t+1)) & \text{if } x^t = x(\cdot, \hat{\ell}, \cdots),\ a^t = a(\hat{\ell}) & \text{and } y^t = y(i) \\ \underline{m}^{k,T} & \text{if } x^t = x(\cdot, \hat{\ell}, \cdots),\ a_{-k}^t = a_{-k}(\hat{\ell}) & \text{and } a_k^t \neq a_k(\hat{\ell}) \\ m^* & & \text{otherwise} \end{cases} \tag{T.1.6}$$

Our next subcase of $\kappa \leq \overline{\kappa}$ is that of $m_i^t = \breve{m}^j$. With the understanding that j' is any element of I not equal to i, we let

$$\mu_i^t(\breve{m}^j, x^t, a^t, y^t) = \begin{cases} \breve{m}^{j'} & \text{if } x^t = x(\cdots, j_\ell, \cdots),\ a^t = \breve{a}^j(j_\ell) & \text{and } y^t = y(j') \\ \nu(\underline{M}(i,t+1)) & \text{if } x^t = x(\cdots, j_\ell, \cdots),\ a^t = \breve{a}^j(j_\ell) & \text{and } y^t = y(i) \\ \underline{m}^{k,T} & \text{if } x^t = x(\cdots, j_\ell, \cdots),\ a_{-k}^t = \breve{a}_{-k}^j(j_\ell) & \text{and } a_k^t \neq \breve{a}_k^j(j_\ell) \\ m^* & & \text{otherwise} \end{cases} \tag{T.1.7}$$

Still assuming $\kappa \leq \overline{\kappa}$ we now deal with the subcase $m_i^t \in \underline{M}(i,t)$. For any $\underline{m}^{i,\tau} \in \underline{M}(i,t)$, we let

$$\mu_i^t(\underline{m}^{i,\tau}, x^t, a^t, y^t) = \begin{cases} \breve{m}^j & \text{if } x^t = x(\cdots, j_\ell, \cdots),\ a^t = \breve{a}^i(i_\ell) & \text{and } y^t = y(j) \\ \nu(\underline{M}(i,t+1)) & \text{if } x^t = x(\cdots, j_\ell, \cdots),\ a^t = \breve{a}^i(i_\ell) & \text{and } y^t = y(i) \\ \underline{m}^{k,T} & \text{if } x^t = x(\cdots, j_\ell, \cdots),\ a_{-k}^t = \breve{a}_{-k}^i(i_\ell) & \text{and } a_k^t \neq \breve{a}_k^i(i_\ell) \\ \underline{m}^{k,T} & \text{if } x^t = x(\cdots, j_\ell, \cdots),\ a_{-k}^t = a_{-k}^i(i_\ell) & \text{and } a_k^t \neq a_k^i(i_\ell) \\ \underline{m}^{i,\tau-1} & \text{if } x^t = x(\cdots, j_\ell, \cdots) & \text{and } a^t = a(i_\ell) \\ m^* & & \text{otherwise} \end{cases} \tag{T.1.8}$$

where we set $\underline{m}^{i,0} = \overline{m}^i$. Notice that player $\langle i,t\rangle$ may need to distinguish between the third and fourth cases of (T.1.8) since clearly they may be generated by different values of the index $k \in I$. To verify that this distinction is always feasible, recall that, by construction (see Definition A.3), $\breve{a}_{-i}(i_\ell)$ differs from $a_{-i}(i_\ell)$ in every component, and that of course $n \geq 4$.

The next subcase of $\kappa \leq \overline{\kappa}$ we consider is that of $m_i^t \in \underline{M}(j,t)$. For any $\underline{m}^{j,\tau} \in \underline{M}(j,t)$, we let

$$\mu_i^t(\underline{m}^{j,\tau}, x^t, a^t, y^t) = \begin{cases} \underline{m}^{j,\tau-1} & \text{if } x^t = x(\cdots, j_\ell, \cdots) & \text{and } a^t = a(j_\ell) \\ \underline{m}^{k,T} & \text{if } x^t = x(\cdots, j_\ell, \cdots),\ a_{-k}^t = a_{-k}(j_\ell) & \text{and } a_k^t \neq a_k(j_\ell) \\ m^* & & \text{otherwise} \end{cases} \tag{T.1.9}$$

where we set $\underline{m}^{j,0} = \overline{m}^j$.

Finally, still assuming that $\kappa \leq \overline{\kappa}$, we consider the case in which $m_i^t = \overline{m}^{k'}$ for some $k' \in I$. We let

$$\mu_i^t(\overline{m}^{k'}, x^t, a^t, y^t) = \begin{cases} \overline{m}^{k'} & \text{if } x^t = x(\cdots, \ell_{k'}, \cdots) & \text{and } a^t = a(\ell_{k'}) \\ \underline{m}^{k,T} & \text{if } x^t = x(\cdots, \ell_{k'}, \cdots),\ a_{-k}^t = a_{-k}(\ell_{k'}) & \text{and } a_k^t \neq a_k(\ell_{k'}) \\ m^* & & \text{otherwise} \end{cases} \tag{T.1.10}$$

T.2. Notation

Point of Notation T.2.1*: Abusing the notation we established for the standard repeated game, we adopt the following notation for continuation payoffs in the dynastic repeated game. Let an assessment (g, μ, Φ) be given.*

[T.2] Throughout the paper we adopt the following notational convention. Given any finite set, we denote by by $\nu(\cdot)$ the uniform probability distribution over the set. So, if B is a finite set, $\nu(B)$ assigns probability $1/\|B\|$ to every element of B.

Recall that we denote by $v_i^t(g,\mu|m_i^t,x^t,\Phi_i^{tB})$ *the continuation payoff to player* $\langle i,t\rangle$ *given the profile* (g,μ)*, after he has received message* m_i^t*, has observed the realization* x^t*, and given that his beliefs over the* $n-1$*-tuple* m_{-i}^t *are* Φ_i^{tB}*. In view of our discussion at the beginning of Section 3, it is clear that the only component of the system of beliefs* Φ *that is relevant to define this continuation payoff is in fact* Φ_i^{tB}*. Our discussion there also implies that the argument* m_i^t *is redundant once* Φ_i^{tB} *has been specified. We keep it in our notation since it helps streamline some of the arguments below.*

We let $v_i^t(g,\mu|m_i^t,x^t,a^t,y^t,\Phi_i^{tE})$ *denote the continuation payoff (viewed from the beginning of period* $t+1$*) to player* $\langle i,t\rangle$ *given the profile* (g,μ)*, after he has received message* m_i^t*, has observed the triple* (x^t,a^t,y^t)*, and given that his beliefs over the* $n-1$*-tuple* m_{-i}^{t+1} *are given by* Φ_i^{tE}*. In view of our discussion at the beginning of Section 3, it is clear that once* Φ_i^{tE} *has been specified, the arguments* (m_i^t,x^t,a^t,y^t) *are redundant in determining the end-of-period continuation payoff to player* $\langle i,t\rangle$*. Whenever this does not cause any ambiguity (about* Φ_i^{tE}*) we will write* $v_i^t(g,\mu|\Phi_i^{tE})$ *instead of* $v_i^t(g,\mu|m_i^t,x^t,a^t,y^t,\Phi_i^{tE})$*.*

As we noted in the text all continuation payoffs clearly depend on δ *as well. To keep notation down this dependence will be omitted whenever possible.*

Point of Notation T.2.2*: We will abuse our notation for* $\Phi_i^{tB}(\cdot)$*,* $\Phi_i^{tE}(\cdot)$ *and* $\Phi_i^{tR}(\cdot)$ *slightly in the following way. We will allow events of interest and conditioning events to appear as arguments of* Φ_i^{tB}*,* Φ_i^{tE} *and* Φ_i^{tR}*, to indicate their probabilities under these distributions.*

So, for instance when we write $\Phi_i^{tB}(m_{-i}^t=(z,\ldots,z)|m_i^t)=c$ *we mean that according to the beginning-of-period beliefs of player* $\langle i,t\rangle$*, after observing* m_i^t*, the probability that* m_{-i}^t *is equal to the* $n-1$*-tuple* $(z,\ldots,z)$ *is equal to* c*.*

Point of Notation T.2.3*: Whenever the profile* (g,μ) *is a profile of completely mixed strategies, the beliefs* $\Phi_i^{tB}(\cdot)$*,* $\Phi_i^{tE}(\cdot)$ *and* $\Phi_i^{tR}(\cdot)$ *are of course entirely determined by what player* $\langle i,t\rangle$ *observes and by* (g,μ) *using Bayes' rule. In this case, we will allow the pair* (g,μ) *to appear as a "conditioning event."*

So, for instance, $\Phi_i^{tB}(m_{-i}^t|m_i^t,\,g,\mu)$ *is the probability of the* $n-1$*-tuple* m_{-i}^t*, after* m_i^t *has been received, obtained from the completely mixed profile* (g,μ) *via Bayes' rule. Events may appear as arguments in this case as well, consistently with our Point of Notation T.2.2 above.*

Moreover, since the completely mixed pair (g,μ) *determines the probabilities of all events, concerning for instance histories, messages of previous cohorts and the like, we will use the notation* Pr *to indicate such probabilities, using the pair* (g,μ) *as a conditioning event.*

So, given any two events L *and* J*, the notation* $\Pr(L|J,g,\mu)$ *will indicate the probability of event* L*, conditional on event* J*, as determined by the completely mixed pair* (g,μ) *via Bayes' rule.*

T.3. A Preliminary Result

As we mentioned before, we work with message spaces that are smaller than the set H^t. We now proceed to show that this is without loss of generality.

Definition T.3.1*: Consider the dynastic repeated game described in full in Section 2. Now consider the dynastic repeated game obtained from this when we restrict the message space of player* $\langle i,t\rangle$ *to be* $M_i^{t+1}\subseteq H^{t+1}$*, with all other details unchanged.*

We call this the restricted dynastic repeated game with message spaces $\{M_i^t\}_{i\in I,t\geq 1}$*. For any given* $\delta\in(0,1)$*,* $\tilde{x}$ *and* $\tilde{y}$*, we denote by* $\mathcal{G}^D(\delta,\tilde{x},\tilde{y},\{M_i^t\}_{i\in I,t\geq 1})$ *the set of SE strategy profiles, while we write* $\mathcal{E}^D(\delta,\tilde{x},\tilde{y},\{M_i^t\}_{i\in I,t\geq 1})$ *for the set of SE payoff profiles of this dynastic repeated game with restricted message spaces.*

Lemma T.3.1*: Let any* $\delta\in(0,1)$*,* $\tilde{x}$ *and* $\tilde{y}$ *be given. Consider now any restricted dynastic repeated game with message spaces* $\{M_i^t\}_{i\in I,t\geq 1}$*. Then* $\mathcal{E}^D(\delta,\tilde{x},\tilde{y},\{M_i^t\}_{i\in I,t\geq 1})\subseteq\mathcal{E}^D(\delta,\tilde{x},\tilde{y})$*.*

Proof: Let a profile $(g^*, \mu^*) \in \mathcal{G}^D(\delta, \tilde{x}, \tilde{y}, \{M_i^t\}_{i\in I, t\geq 1})$ with associated beliefs Φ^* be given. To prove the statement, we proceed to construct a new profile $(g^{**}, \mu^{**}) \in \mathcal{G}^D(\delta, \tilde{x}, \tilde{y})$ and associated beliefs Φ^{**} that are consistent with (g^{**}, μ^{**}), and which gives every player the same payoff as (g^*, μ^*).

Denote a generic element of M_i^t by z_i^t. Since $M_i^t \subseteq H^t$, we can partition H^t into $\|M_i^t\|$ non-empty mutually exclusive exhaustive subsets, and make each of these subsets correspond to an element z_i^t of M_i^t. In other words, we can find a map $\rho_i^t : M_i^t \to 2^{H^t}$ such that $\rho_i^t(z_i^t) \neq \emptyset$ for all $z_i^t \in M_i^t$, $\rho_i^t(z_i^{t\prime}) \cap \rho_i^t(z_i^{t\prime\prime}) = \emptyset$ whenever $z_i^{t\prime} \neq z_i^{t\prime\prime}$, and $\bigcup_{z_i^t \in M_i^t} \rho(z_i^t) = H^t$.

We can now describe how the profile (g^{**}, μ^{**}) is obtained from the given (g^*, μ^*). We deal first with the action stage. For any player $\langle i, t\rangle$, and any $z_i^t \in M_i^t$, set

$$g_i^{t**}(m_i^t, x) = g_i^{t*}(z_i^t, x) \quad \forall\, m_i^t \in \rho_i^t(z_i^t) \tag{T.3.1}$$

At the message stage, for any player $\langle i, t\rangle$, any (z_i^t, x^t, a^t, y^t), any $m_i^t \in \rho_i^t(z_i^t)$, and any $z_i^{t+1} \in \mathrm{Supp}(\mu_i^{t*}(z_i^t, x^t, a^t, y^t))$, set

$$\mu_i^{t**}(m_i^{t+1}|m_i^t, x^t, a^t, y^t) = \frac{1}{\left\|\rho_i^{t+1}(z_i^{t+1})\right\|}\mu_i^{t*}(z_i^{t+1}|z_i^t, x^t, a^t, y^t) \quad \forall\, m_i^{t+1} \in \rho_i^{t+1}(z_i^{t+1}) \tag{T.3.2}$$

Next, we describe Φ^{**}, starting with the beginning-of-period beliefs. For any player $\langle i, t\rangle$, any $z_i^t \in M_i^t$ and any $z_{-i}^t \in M_{-i}^t$, set

$$\Phi_i^{tB**}(m_{-i}^t|m_i^t) = \frac{\Phi_i^{tB*}(z_{-i}^t|z_i^t)}{\Pi_{j\neq i}\left\|\rho_j^t(z_j^t)\right\|} \quad \forall\, m_i^t \in \rho_i^t(z_i^t),\ \ \forall\, m_{-i}^t \in \Pi_{j\neq i}\rho_j^t(z_j^t) \tag{T.3.3}$$

Similarly, concerning the end-of-period beliefs, for any player $\langle i, t\rangle$, any (z_i^t, x^t, a^t, y^t) and any $z_{-i}^{t+1} \in M_{-i}^{t+1}$, set

$$\Phi_i^{tE**}(m_{-i}^{t+1}|m_i^t, x^t, a^t, y^t) = \frac{\Phi_i^{tE*}(z_{-i}^{t+1}|z_i^t, x^t, a^t, y^t)}{\Pi_{j\neq i}\left\|\rho_j^{t+1}(z_j^{t+1})\right\|} \quad \forall\, m_i^t \in \rho_i^t(z_i^t),\ \ \forall\, m_{-i}^{t+1} \in \Pi_{j\neq i}\rho_j^{t+1}(z_j^{t+1}) \tag{T.3.4}$$

Since the profile (g^*, μ^*) is sequentially rational given Φ^*, it is immediate from (T.3.1), (T.3.2), (T.3.3) and (T.3.4) that the profile (g^{**}, μ^{**}) is sequentially rational given Φ^{**}, and we omit further details of the proof of this claim.

Of course, it remains to show that $(g^{**}, \mu^{**}, \Phi^{**})$ is a consistent assessment.

Let $(g_\varepsilon^*, \mu_\varepsilon^*)$ be parameterized completely mixed strategies which converge to (g^*, μ^*) and give rise, in the limit as $\varepsilon \to 0$, to beliefs Φ^* via Bayes' rule.

Given any $\varepsilon > 0$, let $(g_\varepsilon^{**}, \mu_\varepsilon^{**})$ be a profile of completely mixed strategies obtained from $(g_\varepsilon^*, \mu_\varepsilon^*)$ exactly as in (T.3.1) and (T.3.2).

We start by verifying the consistency of the beginning-of-period beliefs. Observe that for any given $z^t = (z_i^t, z_{-i}^t)$, from (T.3.2) we know that whenever $m^t = (m_i^t, m_{-i}^t) \in \Pi_{j\in I}\rho_j^t(z_j^t)$

$$\Pr(m_i^t, m_{-i}^t|g_\varepsilon^{**}, \mu_\varepsilon^{**}) = \frac{\Pr(z_i^t, z_{-i}^t|g_\varepsilon^*, \mu_\varepsilon^*)}{\Pi_{j\in I}\left\|\rho_j^t(z_j^t)\right\|} \tag{T.3.5}$$

Similarly, using (T.3.2) again we know that whenever $m_i^t \in \rho_i^t(z_i^t)$

$$\Pr(m_i^t|g_\varepsilon^{**}, \mu_\varepsilon^{**}) = \frac{\Pr(z_i^t|g_\varepsilon^*, \mu_\varepsilon^*)}{\|\rho_i^t(z_i^t)\|} \tag{T.3.6}$$

Taking the ratio of (T.3.5) and (T.3.6) and taking the limit as $\varepsilon \to 0$ now yields that for any any $z_i^t \in M_i^t$ and any $z_{-i}^t \in M_{-i}^t$

$$\lim_{\varepsilon \to 0} \Phi_i^{tB**}(m_{-i}^t | m_i^t, g_\varepsilon^{**}, \mu_\varepsilon^{**}) = \frac{\Phi_i^{tB*}(z_{-i}^t | z_i^t)}{\Pi_{j \neq i} \left\| \rho_j^t(z_j^t) \right\|} \quad \forall\, m_i^t \in \rho_i^t(z_i^t), \;\; \forall\, m_{-i}^t \in \Pi_{j \neq i} \rho_j^t(z_j^t) \tag{T.3.7}$$

Hence we have shown that the beginning-of-period beliefs as in (T.3.3) are consistent with (g^{**}, μ^{**}).

The proof that the end-of-period beliefs as in (T.3.4) are consistent with (g^{**}, μ^{**}) runs along exactly the same lines, and we omit the details. ■

T.4. Proof of Theorem A.1: Beliefs

Definition T.4.1. *Beginning-of-Period Beliefs: Let k be any element of I, and j be any element of I not equal to i.*

The beginning-of-period beliefs of all players $\langle i \in I, 0\rangle$ are trivial. Of course, all players believe that all other players have received the null message $m_i^0 = \emptyset$.

The beginning-of-period beliefs $\Phi_i^{tB}(m_i^t)$ of any other player $\langle i, t\rangle$, depending on the message he receives from player $\langle i, t-1\rangle$ are as follows[T.3]

$$\begin{array}{lll}
\text{if } m_i^t = m^* & \text{then} & m_{-i}^t = (m^*, \ldots, m^*) \text{ with probability } 1 \\
\text{if } m_i^t = \check{m}^j & \text{then} & \left\{ \begin{array}{ll} m_{-i-j}^t = (\check{m}^j, \ldots, \check{m}^j) & \text{with pr. } 1 \\ m_j^t \in \underline{M}(j,t) & \text{with pr. } 1 \\ \Pr(m_j^t = \underline{m}^{j,\tau}) > 0 & \forall\, \underline{m}^{j,\tau} \in \underline{M}(j,t) \end{array} \right. \\
\text{if } m_i^t = \underline{m}^{j,\tau} & \text{then} & m_{-i}^t = (\underline{m}^{j,\tau}, \ldots, \underline{m}^{j,\tau}) \text{ with probability } 1 \\
\text{if } m_i^t = \underline{m}^{i,\tau} & \text{then} & m_{-i}^t = (\check{m}^i, \ldots, \check{m}^i) \text{ with probability } 1 \\
\text{if } m_i^t = \overline{m}^k & \text{then} & m_{-i}^t = (\overline{m}^k, \ldots, \overline{m}^k) \text{ with probability } 1
\end{array} \tag{T.4.1}$$

Definition T.4.2. *End-of-Period Beliefs: Let k be any element of I, and j be any element of I not equal to i.*

We begin with period $t = 0$. Recall that $m_i^0 = \emptyset$ for all $i \in I$. As before, let also $g^0(m^0, x^0) = (g_1^0(m_1^0, x^0), \ldots, g_n^0(m_n^0, x^0))$, and define $g_{-k}^0(m^0, x^0)$ in the obvious way.

Let $\Phi_i^{0E}(m_i^0, x^0, a^0, y^0)$ be as follows

$$\begin{array}{ll}
\text{if } a^0 = g^0(m^0, x^0) \text{ and } y^0 = y(j) & \text{then } m_{-i-j}^1 = (\check{m}^j, \ldots, \check{m}^j), m_j^1 = \underline{m}^{j,T} \text{ with pr. } 1 \\
\text{if } a^0 = g^0(m^0, x^0) \text{ and } y^0 = y(i) & \text{then } m_{-i}^1 = (\check{m}^i, \ldots, \check{m}^i) \text{ with probability } 1 \\
\text{if } a_{-k}^0 = g_{-k}^0(m^0, x^0) \text{ and } a_k^0 \neq g_k^0(m_k^0, x^0) & \text{then } m_{-i}^1 = (\underline{m}^{k,T}, \ldots, \underline{m}^{k,T}) \text{ with prob. } 1 \\
\text{otherwise} & \quad\;\; m_{-i}^1 = (m^*, \ldots, m^*) \text{ with probability } 1
\end{array} \tag{T.4.2}$$

Our next case is $t \geq 1$ and $x^t = x(\kappa)$ with $\kappa > \overline{\kappa}$. Let $x(\ell_{00}, \ell^*)$ denote the realization of x^t. For any

[T.3] Notice that the second line of (T.4.1) does not fully specify the probability distribution over the component m_j^t of the beliefs of player $\langle i, t\rangle$. For the rest of the argument, what matters is only that all elements of $\underline{M}(j,t)$ have positive probability, and that no message outside this set has positive probability. The distribution can be computed using Bayes' rule from the equilibrium strategies described in Definitions T.1.1 and T.1.2 above. We omit the details for the sake of brevity.

player $\langle i,t\rangle$, let $\Phi_i^{tE}(m_i^t, x(\ell_{00},\ell^*), a^t, y^t)$ be as follows[T.4]

$$
\begin{array}{lll}
\text{if } a^t = a(\ell^*) \text{ and } m_i^t = \breve{m}^j & \text{then} & \begin{cases} m_{-i-j}^{t+1} = (\breve{m}^j,\ldots,\breve{m}^j) & \text{with pr. } 1 \\ m_j^{t+1} \in \underline{M}(j,t) & \text{with pr. } 1 \\ \Pr(m_j^{t+1} = \underline{m}^{j,\tau}) > 0 & \forall\, \underline{m}^{j,\tau} \in \underline{M}(j,t) \end{cases} \\
\text{if } a^t = a(\ell^*) \text{ and } m_i^t = \underline{m}^{j,\tau} & \text{then} & m_{-i}^{t+1} = (\underline{m}^{j,\tau},\ldots,\underline{m}^{j,\tau}) \text{ with probability } 1 \\
\text{if } a^t = a(\ell^*) \text{ and } m_i^t = \underline{m}^{i,\tau} & \text{then} & m_{-i}^{t+1} = (\breve{m}^i,\ldots,\breve{m}^i) \text{ with probability } 1 \\
\text{if } a^t = a(\ell^*) \text{ and } m_i^t = \overline{m}^k & \text{then} & m_{-i}^{t+1} = (\overline{m}^k,\ldots,\overline{m}^k) \text{ with probability } 1 \\
\text{if } a_{-k}^t = a_{-k}(\ell^*) \text{ and } a_k^t \neq a_k(\ell^*) & \text{then} & m_{-i}^{t+1} = (\underline{m}^{k,T},\ldots,\underline{m}^{k,T}) \text{ with probability } 1 \\
\text{otherwise} & & m_{-i}^{t+1} = (m^*,\ldots,m^*) \text{ with probability } 1
\end{array}
\tag{T.4.3}
$$

We divide the case of $t \geq 1$ and $x^t = x(\kappa)$ with $\kappa \leq \overline{\kappa}$ into several subcases, according to which message player $\langle i,t\rangle$ has received. We begin with $m_i^t = m^*$. Let $x(\cdot,\hat{\ell},\cdots)$ denote the realization of x^t. For any player $\langle i,t\rangle$, with the understanding that $\underline{m}^{j,\tau}$ is a generic element of $\underline{M}(j,t+1)$, let $\Phi_i^{tE}(m^*, x(\cdot,\hat{\ell},\cdots), a^t, y^t)$ be as follows

$$
\begin{array}{lll}
\text{if } a^t = a(\hat{\ell}) \text{ and } y^t = y(j) & \text{then} & \begin{cases} m_{-i-j}^{t+1} = (\breve{m}^j,\ldots,\breve{m}^j) \\ m_j^{t+1} = \underline{m}^{j,\tau} \end{cases} \text{ with pr. } \dfrac{1}{\|\underline{M}(j,t+1)\|} \\
\text{if } a^t = a(\hat{\ell}) \text{ and } y^t = y(i) & \text{then} & m_{-i}^{t+1} = (\breve{m}^i,\ldots,\breve{m}^i) \text{ with probability } 1 \\
\text{if } a_{-k}^t = a_{-k}(\hat{\ell}) \text{ and } a_k^t \neq a_k(\hat{\ell}) & \text{then} & m_{-i}^{t+1} = (\underline{m}^{k,T},\ldots,\underline{m}^{k,T}) \text{ with probability } 1 \\
\text{otherwise} & & m_{-i}^{t+1} = (m^*,\ldots,m^*) \text{ with probability } 1
\end{array}
\tag{T.4.4}
$$

The next subcase is that of $m_i^t = \breve{m}^j$. Let $x(\cdots,j_\ell,\cdots)$ denote the realization of x^t. With the understanding that j' is an element of I not equal to i and that $\underline{m}^{j',\tau}$ is a generic element of $\underline{M}(j',t+1)$, let $\Phi_i^{tE}(\breve{m}^j, x(\cdots,j_\ell,\cdots), a^t, y^t)$ be as follows

$$
\begin{array}{lll}
\text{if } a^t = \breve{a}^j(j_\ell) \text{ and } y^t = y(j') & \text{then} & \begin{cases} m_{-i-j}^{t+1} = (\breve{m}^{j'},\ldots,\breve{m}^{j'}) \\ m_{j'}^{t+1} = \underline{m}^{j',\tau} \end{cases} \text{ with pr. } \dfrac{1}{\|\underline{M}(j',t+1)\|} \\
\text{if } a^t = \breve{a}^j(j_\ell) \text{ and } y^t = y(i) & \text{then} & m_{-i}^{t+1} = (\breve{m}^i,\ldots,\breve{m}^i) \text{ with probability } 1 \\
\text{if } a_{-k}^t = \breve{a}_{-k}^j(j_\ell) \text{ and } a_k^t \neq \breve{a}_k^j(j_\ell) & \text{then} & m_{-i}^{t+1} = (\underline{m}^{k,T},\ldots,\underline{m}^{k,T}) \text{ with probability } 1 \\
\text{otherwise} & & m_{-i}^{t+1} = (m^*,\ldots,m^*) \text{ with probability } 1
\end{array}
\tag{T.4.5}
$$

The next subcase is that of $m_i^t = \underline{m}^{i,\tau} \in \underline{M}(i,t)$. Let $x(\cdots,i_\ell,\cdots)$ denote the realization of x^t. With the understanding that $\underline{m}^{j,\tau}$ is a generic element of $\underline{M}(j,t+1)$, let $\Phi_i^{tE}(\underline{m}^{i,\tau}, x(\cdots,i_\ell,\cdots), a^t, y^t)$ be as follows

$$
\begin{array}{lll}
\text{if } a^t = \breve{a}^i(i_\ell) \text{ and } y^t = y(j) & \text{then} & \begin{cases} m_{-i-j}^{t+1} = (\breve{m}^j,\ldots,\breve{m}^j) \\ m_j^{t+1} = \underline{m}^{j,\tau} \end{cases} \text{ with pr. } \dfrac{1}{\|\underline{M}(j,t+1)\|} \\
\text{if } a^t = \breve{a}^i(i_\ell) \text{ and } y^t = y(i) & \text{then} & m_{-i}^{t+1} = (\breve{m}^i,\ldots,\breve{m}^i) \text{ with probability } 1 \\
\text{if } a_{-k}^t = \breve{a}_{-k}^i(i_\ell) \text{ and } a_k^t \neq \breve{a}_k^i(i_\ell) & \text{then} & m_{-i}^{t+1} = (\underline{m}^{k,T},\ldots,\underline{m}^{k,T}) \text{ with probability } 1 \\
\text{if } a_{-k}^t = a_{-k}(i_\ell) \text{ and } a_k^t \neq a_k(i_\ell) & \text{then} & m_{-i}^{t+1} = (\underline{m}^{k,T},\ldots,\underline{m}^{k,T}) \text{ with probability } 1 \\
\text{if } a^t = a(i_\ell) & \text{then} & m_{-i}^{t+1} = (\underline{m}^{i,\tau-1},\ldots,\underline{m}^{i,\tau-1}) \text{ with probability } 1 \\
\text{otherwise} & & m_{-i}^{t+1} = (m^*,\ldots,m^*) \text{ with probability } 1
\end{array}
\tag{T.4.6}
$$

where we set $\underline{m}^{i,0} = \overline{m}^i$.

The next subcase of $t \geq 1$ and $x^t = x(\kappa)$ with $\kappa \leq \overline{\kappa}$ that we consider is that of $m_i^t = \underline{m}^{j,\tau} \in \underline{M}(j,t)$.

[T.4] Similarly to (T.4.1), the first line of (T.4.3) does not fully specify the probability distribution over the component m_j^{t+1} of the beliefs of player $\langle i,t\rangle$. For the rest of the argument, what matters is only that all elements of $\underline{M}(j,t)$ have positive probability, and that no message outside this set has positive probability. The distribution can be computed using Bayes' rule from the equilibrium strategies described in Definitions T.1.1 and T.1.2 above. We omit the details for the sake of brevity.

Let $x(\cdots,j_\ell,\cdots)$ denote the realization of x^t. Let $\Phi_i^{tE}(\underline{m}^{j,\tau},x(\cdots,j_\ell,\cdots),a^t,y^t)$ be as follows

$$\begin{array}{lll} \text{if } a^t=a(j_\ell) & \text{then } m_{-i}^{t+1}=(\underline{m}^{j,\tau-1},\ldots,\underline{m}^{j,\tau-1}) & \text{with probability 1} \\ \text{if } a_{-k}^t=a_{-k}(j_\ell) \text{ and } a_k^t\neq a_k(j_\ell) & \text{then } m_{-i}^{t+1}=(\underline{m}^{k,T},\ldots,\underline{m}^{k,T}) & \text{with probability 1} \\ \text{otherwise} & \quad\; m_{-i}^{t+1}=(m^*,\ldots,m^*) & \text{with probability 1} \end{array} \tag{T.4.7}$$

where we set $\underline{m}^{j,0}=\overline{m}^j$.

The final subcase to consider is that of $m_i^t=\overline{m}^{k'}$ for some $k'\in I$. Let $x(\cdots,\ell_{k'},\cdots)$ denote the realization of x^t. Let $\Phi_i^{tE}(\overline{m}^{k'},x(\cdots,\ell_{k'},\cdots),a^t,y^t)$ be as follows

$$\begin{array}{lll} \text{if } a^t=a(\ell_{k'}) & \text{then } m_{-i}^{t+1}=(\overline{m}^{k'},\ldots\overline{m}^{k'}) & \text{with probability 1} \\ \text{if } a_{-k}^t=a_{-k}(\ell_{k'}) \text{ and } a_k^t\neq a_k(\ell_{k'}) & \text{then } m_{-i}^{t+1}=(\underline{m}^{k,T},\ldots,\underline{m}^{k,T}) & \text{with probability 1} \\ \text{otherwise} & \quad\; m_{-i}^{t+1}=(m^*,\ldots,m^*) & \text{with probability 1} \end{array} \tag{T.4.8}$$

T.5. Proof of Theorem A.1: Sequential Rationality

Definition T.5.1: *Let $\boldsymbol{\mathcal{I}}_i^{tE}$ denote the end-of-period-t collection of information sets that belong to player $\langle i,t\rangle$, with typical element $\mathcal{I}_i^{tE}$.*

It is convenient to partition $\boldsymbol{\mathcal{I}}_i^{tE}$ into mutually disjoint exhaustive subsets on the basis of the associated beliefs of player $\langle i,t\rangle$. The fact that they exhaust $\boldsymbol{\mathcal{I}}_i^{tE}$ can be checked directly from Definition T.4.2 above.

Let $\boldsymbol{\mathcal{I}}_i^{tE}(*)\subset\boldsymbol{\mathcal{I}}_i^{tE}$ be the collection of information sets in which player $\langle i,t\rangle$ believes that m_{-i}^{t+1} is equal to $(m^*,\ldots,m^*)$ with probability one. These beliefs will be denoted by $\Phi_i^{tE}(*)$.

Let $\boldsymbol{\mathcal{I}}_i^{tE}(\smallsmile i)\subset\boldsymbol{\mathcal{I}}_i^{tE}$ be the collection of information sets in which player $\langle i,t\rangle$ believes that m_{-i}^{t+1} is equal to $(\breve{m}^i,\ldots,\breve{m}^i)$ with probability one. These beliefs will be denoted by $\Phi_i^{tE}(\smallsmile i)$.

For every $j\in I$ not equal to i, let $\boldsymbol{\mathcal{I}}_i^{tE}(\smallsmile j,t)\subset\boldsymbol{\mathcal{I}}_i^{tE}$ be the collection of information sets in which player $\langle i,t\rangle$ believes that m_{-i-j}^{t+1} is equal to $(\breve{m}^j,\ldots,\breve{m}^j)$ with probability one, that $\Pr(m_j^{t+1}=\underline{m}^{j,\tau})>0\ \forall\ \underline{m}^{j,\tau}\in\underline{M}(j,t)$, and that $\Pr(m_j^{t+1}\in\underline{M}(j,t))=1$.[T.5] *These beliefs will be denoted by $\Phi_i^{tE}(\smallsmile j,t)$.*

For every $j\in I$ not equal to i, let $\boldsymbol{\mathcal{I}}_i^{tE}(\smallsmile j,t+1)\subset\boldsymbol{\mathcal{I}}_i^{tE}$ be the collection of information sets in which player $\langle i,t\rangle$ believes that m_{-i-j}^{t+1} is equal to $(\breve{m}^j,\ldots,\breve{m}^j)$ with probability one, that $\Pr(m_j^{t+1}=\underline{m}^{j,\tau})=\|\underline{M}(j,t+1)\|^{-1}\ \forall\ \underline{m}^{j,\tau}\in\underline{M}(j,t+1)$. These beliefs will be denoted by $\Phi_i^{tE}(\smallsmile j,t+1)$.

For every $k\in I$, let $\boldsymbol{\mathcal{I}}_i^{tE}(\overline{k})\subset\boldsymbol{\mathcal{I}}_i^{tE}$ be the collection of information sets in which player $\langle i,t\rangle$ believes that m_{-i}^{t+1} is equal to $(\overline{m}^k,\ldots,\overline{m}^k)$ with probability one. These beliefs will be denoted by $\Phi_i^{tE}(\overline{k})$.

For every $k\in I$, and every $\tau=\max\{T-t,1\},\ldots,T$ let $\boldsymbol{\mathcal{I}}_i^{tE}(\underline{k},\tau)\subset\boldsymbol{\mathcal{I}}_i^{tE}$ be the collection of information sets in which player $\langle i,t\rangle$ believes that m_{-i}^{t+1} is equal to $(\underline{m}^{k,\tau},\ldots,\underline{m}^{k,\tau})$ with probability one. These beliefs will be denoted by $\Phi_i^{tE}(\underline{k},\tau)$.

Definition T.5.2: *Let the strategy profile (g,μ) described in Definitions T.1.1 and T.1.2 be given. Fix a period t and an n-tuple of messages $m^{t+1}=(m_1^{t+1},\ldots,m_n^{t+1})$, with $m_k^{t+1}\in M_k^{t+1}$ for every $k\in I$.*

Clearly, the profile (g,μ) together with m^{t+1} uniquely determine a probability distribution over action profiles over all future periods, beginning with $t+1$.

Therefore, we can define the expected discounted (from the beginning of period $t+1$) payoff to player $\langle i,t\rangle$, given (g,μ) and m^{t+1} in the obvious way. This will be denoted by $\ddot{v}_i^t(m^{t+1})$. Moreover, since they play a special role in some of the computations that follow, we reserve two pieces of notation for two particular instances of m^{t+1}. The expression $\ddot{v}_i^t(*)$ stands for $\ddot{v}_i^t(m^{t+1})$ when $m^{t+1}=(m^*,\ldots,m^*)$. Moreover, for any $k\in I$, the expression $\ddot{v}_i^t(k,\tau)$ stands for $\ddot{v}_i^t(m^{t+1})$ when $m_{-k}^{t+1}=(\breve{m}^k,\ldots\breve{m}^k)$ and $m_k^{t+1}=\underline{m}^{k,\tau}\in\underline{M}(k,t+1)$.

[T.5] See footnote T.4 above.

Lemma T.5.1: *For any $i \in I$, any $k \in I$, any t, and any $\tau = \max\{T-t, 1\}, \ldots, T$, we have that*

$$\ddot{v}_i^t(*) \;=\; \frac{(1-\delta)\left[q\,\hat{\hat{v}}_i \,+\, (1-q)\,z_i\right] \,+\, \delta\,q\,v_i^*}{1\,-\,\delta\,(1\,-\,q)} \tag{T.5.1}$$

and

$$\ddot{v}_i^t(k,\tau) \;=\; \frac{(1-\delta)\left[q\,\breve{u}_i^k \,+\, (1-q)\,z_i\right] \,+\, \delta\,q\,v_i^*}{1\,-\,\delta\,(1\,-\,q)} \tag{T.5.2}$$

where $\ddot{v}_i^t(*)$ and $\ddot{v}_i^t(k,\tau)$ are as in Definition T.5.2, $\hat{\hat{v}}_i$ is as in (A.7), z_i is as in Remark A.4, v_i^* is as in the statement of the Theorem, and $\breve{u}_i^k$ is as in (A.4).

Proof: Assume first that $t \geq T$. Using Definitions T.1.1 and T.1.2 we can write $\ddot{v}_i^t(*)$ and $\ddot{v}_i^t(k,\tau)$ recursively as

$$\ddot{v}_i^t(*) = q\left\{(1-\delta)\hat{\hat{v}}_i + \delta\left[(1-\eta)\ddot{v}_i^{t+1}(*) + \frac{\eta}{n}\sum_{k'\in I}\sum_{\tau=1}^{T}\frac{\ddot{v}_i^{t+1}(k',\tau)}{T}\right]\right\} + (1-q)\left[(1-\delta)z_i + \delta\ddot{v}_i^{t+1}(*)\right] \tag{T.5.3}$$

and

$$\ddot{v}_i^t(k,\tau) = q\left\{(1-\delta)\breve{u}_i^k + \delta\left[(1-\eta)\ddot{v}_i^{t+1}(*) + \frac{\eta}{n}\sum_{k'\in I}\sum_{\tau=1}^{T}\frac{\ddot{v}_i^{t+1}(k',\tau)}{T}\right]\right\} + (1-q)\left[(1-\delta)z_i + \delta\ddot{v}_i^{t+1}(k,\tau)\right] \tag{T.5.4}$$

Since the strategy profile (g,μ) described in Definitions T.1.1 and T.1.2 is stationary for $t \geq T$, we immediately have that $\ddot{v}_i^t(*) = \ddot{v}_i^{t+1}(*)$ and, for any $k \in I$ and any $\tau = 1, \ldots, T$, $\ddot{v}_i^t(k,\tau) = \ddot{v}_i^{t+1}(k,\tau)$. Hence we can solve (T.5.3) and (T.5.4) simultaneously for the $NT+1$ variables $\ddot{v}_i^t(*)$ and $\ddot{v}_i^t(k,\tau)$ ($k \in I$ and $\tau = 1, \ldots, T$). Using (A.8) this immediately gives (T.5.1) and (T.5.2), as required.

Proceeding by induction backwards from $t = T$, it is also immediate to verify that the statement holds for any $t < T$. The details are omitted for the sake of brevity. ■

Lemma T.5.2: *Let the strategy profile (g,μ) and system of beliefs Φ described in Definitions T.1.1, T.1.2, T.4.1 and T.4.2 be given. Then the end-of-period continuation payoffs for any player $\langle i,t\rangle$ (discounted as of the beginning of period $t+1$) at any information set $\mathcal{I}_i^t \in \boldsymbol{\mathcal{I}}_i^{tE}$ (as categorized in Definition T.5.1) are as follows.*[T.6]

$$v_i^t(g,\mu|\Phi_i^{tE}(*)) = \frac{(1-\delta)\left[q\,\hat{\hat{v}}_i \,+\, (1-q)\,z_i\right] \,+\, \delta\,q\,v_i^*}{1\,-\,\delta\,(1\,-\,q)} \tag{T.5.5}$$

$$v_i^t(g,\mu|\Phi_i^{tE}(\smile i)) = \frac{(1-\delta)\left[q\,\breve{u}_i^i \,+\, (1-q)\,z_i\right] \,+\, \delta\,q\,v_i^*}{1\,-\,\delta\,(1\,-\,q)} \tag{T.5.6}$$

$$v_i^t(g,\mu|\Phi_i^{tE}(\smile j,t)) = v_i^t(g,\mu|\Phi_i^{tE}(\smile j,t+1)) = \frac{(1-\delta)\left[q\,\breve{u}_i^j \,+\, (1-q)\,z_i\right] \,+\, \delta\,q\,v_i^*}{1\,-\,\delta\,(1\,-\,q)} \quad \forall j \neq i \tag{T.5.7}$$

[T.6] See our Point of Notation T.2.1 above.

$$v_i^t(g,\mu|\Phi_i^{tE}(\overline{k})) = q\,\overline{v}_i^k \;+\; (1-q)z_i \quad \forall\, k \in I \tag{T.5.8}$$

$$\begin{aligned} v_i^t(g,\mu|\Phi_i^{tE}(\underline{k},\tau)) = &\left[1-\left(\frac{\delta q}{1-\delta(1-q)}\right)^{\tau}\right]\left[q\underline{\omega}_i^k + (1-q)z_i\right] + \\ &\left(\frac{\delta q}{1-\delta(1-q)}\right)^{\tau}\left[q\overline{v}_i^k + (1-q)z_i\right] \quad \forall\, k\in I \quad \forall\, \tau = \max\{T-t,1\},\ldots,T \end{aligned} \tag{T.5.9}$$

where $\hat{v}_i$ is as in (A.7), z_i is as in Remark A.4, v_i^* is as in the statement of the Theorem, $\breve{u}_i^k$ is as in (A.4), and $\underline{\omega}_i^k$ is as in (A.3).

Proof: Equations (T.5.5), (T.5.6) and (T.5.7) are a direct consequence of Definition T.5.1 and Lemma T.5.1.

Equation (T.5.8) follows directly from Definition T.5.1 and the description of the profile (g,μ) in Definitions T.1.1 and T.1.2.

Using the notation established in Definition T.5.2, consider the quantity $\ddot{v}_i^t(\underline{m}^{k,\tau},\ldots,\underline{m}^{k,\tau})$. Given the strategies described in Definitions T.1.1 and T.1.2 it is evident that this quantity does not depend on t. Therefore, for any $k\in I$ and $\tau = \max\{T-t,1\},\ldots,T$, we can let $\ddot{v}_i(\underline{k},\tau) = \ddot{v}_i^t(\underline{m}^{k,\tau},\ldots,\underline{m}^{k,\tau})$, for all t. Clearly, using Definition T.5.1, we have that for all k, τ and t, $v_i^t(g,\mu|\Phi_i^{tE}(\underline{k},\tau)) = \ddot{v}_i(\underline{k},\tau)$.

From the description of (g,μ) in Definitions T.1.1 and T.1.2, for any $k\in I$ and for any $\tau = 2,\ldots,T$, the quantity $\ddot{v}_i(\underline{k},\tau)$ obeys a difference equation as follows.

$$\ddot{v}_i(\underline{k},\tau) = q\left[(1-\delta)\underline{\omega}_i^k + \delta\ddot{v}_i(\underline{k},\tau-1)\right] + (1-q)\left[(1-\delta)z_i + \delta\ddot{v}_i(\underline{k},\tau)\right] \tag{T.5.10}$$

Using again Definitions T.1.1 and T.1.2, the terminal condition for (T.5.10) is

$$\ddot{v}_i(\underline{k},1) = q\left[(1-\delta)\underline{\omega}_i^k + \delta[q\underline{v}_i^k + (1-q)z_i]\right] + (1-q)\left[(1-\delta)z_i + \delta\ddot{v}_i(\underline{k},1)\right] \tag{T.5.11}$$

Solving (T.5.10) and imposing the terminal condition (T.5.11) now yields (T.5.9), as required. ■

Purely for expositional convenience, before completing the proof of sequential rationality at the message stage, we now proceed with the argument that establishes sequential rationality at the action stage.

Definition T.5.3*: Recall that at the action stage, player $\langle i,t\rangle$ chooses an action after having received a message m_i^t and having observed a realization x^t of the randomization device $\tilde{x}^t$.*

Let $\boldsymbol{\mathcal{I}}_i^{tB}$ denote period-t action-stage collection of information sets that belong to player $\langle i,t\rangle$, with typical element $\mathcal{I}_i^{tB}$. Clearly, each element of $\boldsymbol{\mathcal{I}}_i^{tB}$ is identified by a pair (m_i^t,x^t).

It is convenient to partition $\boldsymbol{\mathcal{I}}_i^{tB}$ into mutually disjoint exhaustive subsets. The fact that they exhaust $\boldsymbol{\mathcal{I}}_i^{tB}$ can be checked directly from Definition T.4.1 above.

Let $\boldsymbol{\mathcal{I}}_i^{tB}(*) \subset \boldsymbol{\mathcal{I}}_i^{tB}$ be the collection of information sets in which player $\langle i,t\rangle$ believes that m_{-i}^t is equal to $(m^*,\ldots,m^*)$ with probability one.[T.7] *These beliefs will be denoted by $\Phi_i^{tB}(*)$.*

Let $\boldsymbol{\mathcal{I}}_i^{tB}(\smile i) \subset \boldsymbol{\mathcal{I}}_i^{tB}$ be the collection of information sets in which player $\langle i,t\rangle$ believes that m_{-i}^t is equal to $(\breve{m}^i,\ldots,\breve{m}^i)$ with probability one. These beliefs will be denoted by $\Phi_i^{tB}(\smile i)$.

For every $j\in I$ not equal to i, let $\boldsymbol{\mathcal{I}}_i^{tB}(\smile j) \subset \boldsymbol{\mathcal{I}}_i^{tB}$ be the collection of information sets in which player $\langle i,t\rangle$ believes that m_{-i-j}^t is equal to $(\breve{m}^j,\ldots,\breve{m}^j)$ with probability one, that $\Pr(m_j^t = \underline{m}^{j,\tau}) > 0\ \forall\ \underline{m}^{j,\tau} \in \underline{M}(j,t)$, and that $\Pr(m_j^t \in \underline{M}(j,t)) = 1$.[T.8] *These beliefs will be denoted by $\Phi_i^{tB}(\smile j)$.*

[T.7] In the interest of brevity, we avoid an explicit distinction between the $t=0$ players and all others. What follows can be interpreted as applying to all players re-defining m_i^0 to be equal to m^* for players $\langle i\in I,0\rangle$.

[T.8] See footnote T.3.

For every $j \in I$ not equal to i, and every $\tau = \max\{T-t+1, 1\}, \ldots, T$ let $\boldsymbol{\mathcal{I}}_i^{tB}(\underline{j}, \tau) \subset \boldsymbol{\mathcal{I}}_i^{tB}$ be the collection of information sets in which player $\langle i, t\rangle$ believes that m_{-i}^t is equal to $(\underline{m}^{j,\tau}, \ldots, \underline{m}^{j,\tau})$ with probability one. These beliefs will be denoted by $\Phi_i^{tB}(\underline{j}, \tau)$.

For every $k \in I$, let $\boldsymbol{\mathcal{I}}_i^{tB}(\overline{k}) \subset \boldsymbol{\mathcal{I}}_i^{tB}$ be the collection of information sets in which player $\langle i, t\rangle$ believes that m_{-i}^t is equal to $(\overline{m}^k, \ldots, \overline{m}^k)$ with probability one. These beliefs will be denoted by $\Phi_i^{tE}(\overline{k})$.

Lemma T.5.3: *There exists a $\underline{\delta} \in (0,1)$ such that whenever $\delta > \underline{\delta}$ the action-stage strategies described in Definition T.1.1 are sequentially rational given the beliefs described in Definition T.4.1 for every player $\langle i, t\rangle$.*[T.9]

Proof: Consider any information set $\mathcal{I}_i^{tB} \in \{\boldsymbol{\mathcal{I}}_i^{tB}(*) \cup \boldsymbol{\mathcal{I}}_i^{tB}(\smallsmile i) \cup \boldsymbol{\mathcal{I}}_i^{tB}(\smallsmile j)\}$.[T.10]

Using Definition T.1.1, Lemma T.5.2 and Definition T.5.3, it is immediate to check that, as $\delta \to 1$, the limit expected continuation payoff to player $\langle i, t\rangle$ from following the action-stage strategies described in Definition T.1.1 at any of these information sets is

$$v_i^* \;=\; q\hat{v}_i + (1-q)z_i \tag{T.5.12}$$

In the same way, it can be checked that, as $\delta \to 1$, the limit expected continuation payoff to player $\langle i, t\rangle$ from deviating at any of these information sets is

$$q\overline{v}_i^i + (1-q)z_i \tag{T.5.13}$$

Since by assumption $\hat{v}_i > \overline{v}_i^i$ this is of course sufficient to prove our claim for any information set $\mathcal{I}_i^{tB} \in \{\boldsymbol{\mathcal{I}}_i^{tB}(*) \cup \boldsymbol{\mathcal{I}}_i^{tB}(\smallsmile i) \cup \boldsymbol{\mathcal{I}}_i^{tB}(\smallsmile j)\}$.

Now consider any information set $\mathcal{I}_i^{tB}$ either in $\boldsymbol{\mathcal{I}}_i^{tB}(\underline{j}, \tau)$ or in $\boldsymbol{\mathcal{I}}_i^{tB}(\bar{j})$ (with $j \neq i$).

Using Definition T.1.1, Lemma T.5.2 and Definition T.5.3, it is immediate to check that, as $\delta \to 1$, the limit expected continuation payoff to player $\langle i, t\rangle$ from following the action-stage strategies described in Definition T.1.1 at any of these information sets is

$$q\overline{v}_i^j + (1-q)z_i \tag{T.5.14}$$

In the same way, it can be checked that, as $\delta \to 1$, the limit expected continuation payoff to player $\langle i, t\rangle$ from deviating at any of these information sets is exactly as in (T.5.13).

Since by assumption for any $j \neq i$ we have that $\overline{v}_i^j > \overline{v}_i^i$ this is of course sufficient to prove our claim for any of these information sets.

To conclude the proof of the lemma, we now consider any information set $\mathcal{I}_i^{tB} \in \boldsymbol{\mathcal{I}}_i^{tB}(\bar{i})$. Using Definition T.1.1, Lemma T.5.2 and Definition T.5.3, it can be checked that the expected continuation payoff to player $\langle i, t\rangle$ from following the action-stage strategies described in Definition T.1.1 at any of these information sets is bounded below by

$$(1-\delta)\underline{u}_i + \delta\left[q\overline{v}_i^i + (1-q)z_i\right] \tag{T.5.15}$$

[T.9] It should be understood that we are, for now, taking it as given that each player $\langle i, t\rangle$ follows the prescriptions of the message-stage strategies described in Definition T.1.2. Of course, we have not demonstrated yet that this is in fact sequentially rational given the beliefs described in Definition T.4.2. We will come back to this immediately after the current lemma is proved.

[T.10] See Definition T.5.3.

In the same way it can be readily seen that the expected continuation payoff to player $\langle i,t\rangle$ from deviating at any of these information sets is bounded above by

$$(1-\delta)\overline{u}_i + \delta\left\{\left[1-\left(\frac{\delta q}{1-\delta(1-q)}\right)^T\right]\left[q\underline{\omega}_i^i + (1-q)z_i\right] + \left(\frac{\delta q}{1-\delta(1-q)}\right)^T\left[q\overline{v}_i^i + (1-q)z_i\right]\right\} \tag{T.5.16}$$

The difference given by (T.5.15) minus (T.5.16) can be written as

$$(1-\delta)\left\{\frac{\delta q\left[1-\left(\frac{\delta q}{1-\delta(1-q)}\right)^T\right](\overline{v}_i^i-\underline{\omega}_i^i)}{(1-\delta)} - (\overline{u}_i-\underline{u}_i)\right\} \tag{T.5.17}$$

Consider now the term inside the curly brackets in (T.5.17). We have that

$$\lim_{\delta\to 1}\frac{\delta q\left[1-\left(\frac{\delta q}{1-\delta(1-q)}\right)^T\right](\overline{v}_i^i-\underline{\omega}_i^i)}{(1-\delta)} - (\overline{u}_i-\underline{u}_i) = T(\overline{v}_i^i-\underline{\omega}_i^i) - (\overline{u}_i-\underline{u}_i) \tag{T.5.18}$$

Using (A.11), we know that the quantity on the right-hand side of (T.5.18) is strictly positive. Hence we can conclude our claim is valid at any information set $\mathcal{I}_i^{tB} \in \mathbb{I}_i^{tB}(\bar{i})$. ■

Lemma T.5.4: *Consider the notation we established in Definition T.5.2. For any given t and $\tau = \max\{T-t,1\},\ldots,T$ let $\ddot{v}_i^t(m,\underline{m}^{i,\tau})$ denote $\ddot{v}_i^t(m^{t+1})$ when the vector m^{t+1} has the i-th component equal to a generic $m \in M_i^{t+1}$ and $m_{-i}^{t+1} = (\underline{m}^{i,\tau},\ldots,\underline{m}^{i,\tau})$. As in the proof of Lemma T.5.2, let $\ddot{v}_i(\underline{i},\tau) = \ddot{v}_i^t(\underline{m}^{i,\tau},\ldots,\underline{m}^{i,\tau})$.*

Then there exists a $\underline{\delta} \in (0,1)$ such that whenever $\delta > \underline{\delta}$ for every player $\langle i,t\rangle$, for every $m \in M_i^{t+1}$, and for every $\tau = \max\{T-t,1\},\ldots,T$

$$\ddot{v}_i(\underline{i},\tau) \;\geq\; \ddot{v}_i^t(m,\underline{m}^{i,\tau}) \tag{T.5.19}$$

Proof: We prove the claim for the case $t \geq T$. The treatment of $t < T$ has some completely non-essential complications due to the fact that the players' message spaces increase in size for the first T periods. The details are are omitted for the sake of brevity.

We now introduce a new random random variable $\tilde{w}$, independent of $\tilde{x}$ and $\tilde{y}$ (see Definitions A.7 and A.8), and uniformly distributed over the finite set $\{1,\ldots,T\}$. This will be used in the rest of the proof of the lemma to keep track of the "private" randomization across messages that members of dynasty i may be required to perform (see Definition T.1.2). Just as we did for the action-stage and the message-stage randomization devices, we consider countably many independent "copies" of $\tilde{w}$, one for each time period, denoted by $\tilde{w}^t$, with typical realization w^t.

To keep track of all "future randomness" looking ahead for $t' = 1,2,\ldots$ periods from t, it will also be convenient to define the random vectors $\tilde{s}^{t,t'}$

$$\tilde{s}^{t,t'} = [(\tilde{x}^{t+1},\tilde{y}^{t+1},\tilde{w}^{t+1}),\ldots,(\tilde{x}^{t+t'},\tilde{y}^{t+t'},\tilde{w}^{t+t'})] \tag{T.5.20}$$

A typical realization of $\tilde{s}^{t,t'}$ will be denoted by $s^{t,t'} = [(x^{t+1},y^{t+1},w^{t+1}),\ldots,(x^{t+t'},y^{t+t'},w^{t+t'})]$. The set of all possible realizations of $\tilde{s}^{t,t'}$ (which obviously does not depend on t) is denoted by $S^{t'}$.

Recall that the profile (g,μ) described in Definitions T.1.1 and T.1.2 is taken as given throughout. Now suppose that in period t, player $\langle i,t\rangle$ sends a generic message $m \in M_i^{t+1}$ and that $m_{-i}^{t+1} = (\underline{m}^{i,\tau},\ldots,\underline{m}^{i,\tau})$. Then, given any realization $s^{t,t'}$ we can compute the actual action profile played by all players $\langle k \in I, t+t'\rangle$. This will be denoted by $\mathbf{a}^{t+t'}(m,\underline{m}^{i,\tau},s^{t,t'})$. Similarly, we can compute the profile of messages $m_{-i}^{t+t'}$ received by all players $\langle j \neq i, t+t'\rangle$. This $n-1$-tuple will be denoted by $\mathbf{m}^{t+t'}(m,\underline{m}^{i,\tau},s^{t,t'})$.

Recall that the messages received by all time-$t+t'$ players are the result of choices and random draws that take place on or before period $t+t'-1$. Therefore it is clear that if we are given two realizations $\hat{s}^{t,t'}$ $= [s^{t,t'-1},(\hat{x}^{t+t'},\hat{y}^{t+t'},\hat{w}^{t+t'})]$ and $\hat{\hat{s}}^{t,t'} = [s^{t,t'-1},(\hat{\hat{x}}^{t+t'},\hat{\hat{y}}^{t+t'},\hat{\hat{w}}^{t+t'})]$, then it must be that

$$\mathbf{m}^{t+t'}(m,\underline{m}^{i,\tau},\hat{s}^{t,t'}) = \mathbf{m}^{t+t'}(m,\underline{m}^{i,\tau},\hat{\hat{s}}^{t,t'}) \tag{T.5.21}$$

Notice next that from the description of the profile (g,μ) in Definitions T.1.1 and T.1.2 it is also immediate to check that for any t', any $m \in M_i^{t+1}$ and any realization $s^{t,t'}$ the message profile $\mathbf{m}^{t+t'}(m,\underline{m}^{i,\tau},s^{t,t'})$ can only take one out of two possible forms. Either we have $\mathbf{m}^{t+t'}(m,\underline{m}^{i,\tau},s^{t,t'}) = (\overline{m}^i,\ldots,\overline{m}^i)$ or it must be that $\mathbf{m}^{t+t'}(m,\underline{m}^{i,\tau},s^{t,t'}) = (\underline{m}^{i,\tau'},\ldots,\underline{m}^{i,\tau'})$ for some $\tau' = 1,\ldots,T$.

Lastly, notice that, given an arbitrary message $m \in M_i^{t+1}$ we can write

$$\ddot{v}_i^t(m,\underline{m}^{i,\tau}) = (1-\delta)\sum_{t'=1}^{\infty}\delta^{t'-1}\sum_{s^{t,t'}\in S^{t'}}\Pr(\tilde{s}^{t,t'} = s^{t,t'})u_i[\mathbf{a}^{t+t'}(m,\underline{m}^{i,\tau},s^{t,t'})] \tag{T.5.22}$$

Since the strategies described in Definitions T.1.1 and T.1.2 are stationary for $t \geq T$, and the distribution of $\tilde{s}^{t,t'}$ is independent of t, it is evident from (T.5.22) that $\ddot{v}_i^t(m,\underline{m}^{i,\tau})$ does not depend on t. From now on we drop the superscript and write $\ddot{v}_i(m,\underline{m}^{i,\tau})$.

We now proceed with the proof of inequality (T.5.19) of the statement of the lemma. In order to do so, from now on we fix a particular $t=\hat{t}$, $m=\hat{m}$ and $\tau=\hat{\tau}$, and we prove (T.5.19) for these fixed values of t, m and τ. Since the lower bound on δ that we will find will clearly not depend on t, and since there are finitely many values that m and τ can take, this will be sufficient to prove the claim.

Inequality (T.5.19) in the statement of the lemma is trivially satisfied (as an equality) if $m = \underline{m}^{i,\tau}$. From now on assume that $\hat{m}$ and $\hat{\tau}$ are such that $\hat{m} \neq \underline{m}^{i,\hat{\tau}}$.

Given any $t' = 1,2,\ldots$, we now partition the set of realizations $S^{t'}$ into five disjoint exhaustive subsets; $S_1^{t'}$, $S_2^{t'}$, $S_3^{t'}$, $S_4^{t'}$ and $S_5^{t'}$. This will allow us to decompose the right-hand side of (T.5.22) in a way that will make possible the comparison with (a similar decomposition of) the left-hand side of (T.5.19) as required to prove the lemma.

Let

$$S_1^{t'} \;=\; \{s^{\hat{t},t'} \mid \mathbf{m}^{\hat{t}+t'}(\underline{m}^{i,\hat{\tau}},\underline{m}^{i,\hat{\tau}},s^{\hat{t},t'}) \;=\; (\underline{m}^{i,\tau'},\ldots,\underline{m}^{i,\tau'}) \text{ for some } \tau'=1,\ldots,\hat{\tau}\} \tag{T.5.23}$$

and notice that if $t' \leq \hat{\tau}$ then $S_1^{t'} = S^{t'}$.

Assume now that $t' > \hat{\tau}$ and let

$$\begin{aligned} S_2^{t'} = \{s^{\hat{t},t'} \mid\; & \mathbf{m}^{\hat{t}+t'}(\hat{m},\underline{m}^{i,\hat{\tau}},s^{\hat{t},t'}) = (\overline{m}^i,\ldots,\overline{m}^i) \text{ and} \\ & u_i(\mathbf{a}^{\hat{t}+t'}(\hat{m},\underline{m}^{i,\hat{\tau}},s^{\hat{t},t'})) \leq u_i(\mathbf{a}^{\hat{t}+t'}(\underline{m}^{i,\hat{\tau}},\underline{m}^{i,\hat{\tau}},s^{\hat{t},t'}))\} \end{aligned} \tag{T.5.24}$$

and

$$\begin{aligned} S_3^{t'} = \{s^{\hat{t},t'} \mid\; & \mathbf{m}^{\hat{t}+t'}(\hat{m},\underline{m}^{i,\hat{\tau}},s^{\hat{t},t'}) = (\overline{m}^i,\ldots,\overline{m}^i) \text{ and} \\ & u_i(\mathbf{a}^{\hat{t}+t'}(\hat{m},\underline{m}^{i,\hat{\tau}},s^{\hat{t},t'})) > u_i(\mathbf{a}^{\hat{t}+t'}(\underline{m}^{i,\hat{\tau}},\underline{m}^{i,\hat{\tau}},s^{\hat{t},t'}))\} \end{aligned} \tag{T.5.25}$$

Notice that if the first condition in (T.5.24) holds, then $\mathbf{m}^{\hat{t}+t'}(\underline{m}^{i,\hat{\tau}}, \underline{m}^{i,\hat{\tau}}, s^{\hat{t},t'}) = (\overline{m}^i, \ldots, \overline{m}^i)$. Therefore, $S_1^{t'}$ and $S_2^{t'}$ and $S_3^{t'}$ are disjoint.

Next, let any $s^{\hat{t},t''} \in S_3^{t''}$ with $t'' < t'$ be given and define

$$S_4^{t'}(s^{\hat{t},t''}) = \{s^{\hat{t},t'} \mid s^{\hat{t},t'} = (s^{\hat{t},t''}, s^{t'',t'}) \text{ for some } s^{t'',t'} \text{ and } \|\{t \in (t''+1, \ldots, t'-1) \mid x^t = x(\kappa) \text{ with } \kappa \leq \overline{\kappa}\}\| \leq T-1\} \tag{T.5.26}$$

Now let

$$S_4^{t'} = \bigcup_{\substack{t''<t' \\ s^{\hat{t},t''} \in S_3^{t''}}} S_4^{t'}(s^{\hat{t},t''}) \tag{T.5.27}$$

From the strategies described in Definitions T.1.1 and T.1.2 it can be checked that if $s^{\hat{t},t'} \in S_4^{t'}$ then $\mathbf{m}^{\hat{t}+t'}(\hat{m}, \underline{m}^{i,\hat{\tau}}, s^{\hat{t},t'}) = (\underline{m}^{i,\tau'}, \ldots, \underline{m}^{i,\tau'})$ for some τ' and $\mathbf{m}^{\hat{t}+t'}(\underline{m}^{i,\hat{\tau}}, \underline{m}^{i,\hat{\tau}}, s^{\hat{t},t'}) = (\overline{m}^i, \ldots, \overline{m}^i)$. Therefore, it is clear that $S_4^{t'}$ is disjoint from $S_1^{t'}$, $S_2^{t'}$ and $S_3^{t'}$.

The last set in the partition of $S^{t'}$ is defined as the residual of the previous four.

$$S_5^{t'} = S^{t'} / \{S_1^{t'} \cup S_2^{t'} \cup S_3^{t'} \cup S_4^{t'}\} \tag{T.5.28}$$

Using (T.5.22), we can now proceed to compare the two sides of inequality (T.5.19) of the statement of the lemma for the five distinct (conditional) cases $s^{\hat{t},t'} \in S_1^{t'}$ through $s^{\hat{t},t'} \in S_5^{t'}$. Notice first of all that when $s^{\hat{t},t'} \in S_2^{t'}$, we know immediately from (T.5.24) that there is nothing to prove.

We begin with $s^{\hat{t},t'} \in S_1^{t'}$. Notice first of all that if we fix any $\overline{s}^{\hat{t},t'} \in S_1^{t'}$, then it follows from (T.5.21) and (T.5.23) that any $s^{\hat{t},t'}$ of the form $[\overline{s}^{\hat{t},t'-1}, s^{t'-1,t'}]$ (where $\overline{s}^{\hat{t},t'-1}$ are the first $t'-1$ triples of $\overline{s}^{\hat{t},t'}$) is in fact in $S_1^{t'}$.

Using, (T.5.23) and Definitions A.1, T.1.1 and T.1.2 we get

$$\begin{aligned} &\sum_{s^{t'-1,t'} \in S^1} \Pr(\tilde{s}^{t'-1,t'} = s^{t'-1,t'})\, u_i(\mathbf{a}^{\hat{t}+t'}(\underline{m}^{i,\hat{\tau}}, \underline{m}^{i,\hat{\tau}}, [\overline{s}^{\hat{t},t'-1}, s^{t'-1,t'}])) = q\underline{\omega}_i^i + (1-q)z_i \geq \\ &\qquad \sum_{s^{t'-1,t'} \in S^1} \Pr(\tilde{s}^{t'-1,t'} = s^{t'-1,t'})\, u_i(\mathbf{a}^{\hat{t}+t'}(\hat{m}, \underline{m}^{i,\hat{\tau}}, [\overline{s}^{\hat{t},t'-1}, s^{t'-1,t'}])) \end{aligned} \tag{T.5.29}$$

Therefore, since the $\overline{s}^{\hat{t},t'}$ that we fixed is an arbitrary element of $S_1^{t'}$, we can now conclude that

$$\sum_{s^{\hat{t},t'} \in S_1^{t'}} \Pr(\tilde{s}^{\hat{t},t'} = s^{\hat{t},t'})\, u_i(\mathbf{a}^{\hat{t},t'}(\underline{m}^{i,\hat{\tau}}, \underline{m}^{i,\hat{\tau}}, s^{\hat{t},t'})) \geq \sum_{s^{\hat{t},t'} \in S_1^{t'}} \Pr(\tilde{s}^{\hat{t},t'} = s^{\hat{t},t'})\, u_i(\mathbf{a}^{\hat{t},t'}(\hat{m}, \underline{m}^{i,\hat{\tau}}, s^{\hat{t},t'})) \tag{T.5.30}$$

Now fix any $\overline{s}^{\hat{t},t'} \in S_3^{t'}$. Using, (T.5.25), (T.5.26) and (T.5.27), and Definitions T.1.1 and T.1.2 we get that the difference given by

$$\begin{aligned} &\Pr(\tilde{s}^{\hat{t},t'} = \overline{s}^{\hat{t},t'}) u_i(\mathbf{a}^{\hat{t}+t'}(\underline{m}^{i,\hat{\tau}}, \underline{m}^{i,\hat{\tau}}, \overline{s}^{\hat{t},t'})) + \\ &\qquad \sum_{t''=t'+1}^{\infty} \delta^{(t''-t')} \sum_{s^{\hat{t},t''} \in S_4^{t''}(\overline{s}^{\hat{t},t'})} \Pr(\tilde{s}^{\hat{t},t''} = \overline{s}^{\hat{t},t''}) u_i(\mathbf{a}^{\hat{t}+t''}(\underline{m}^{i,\hat{\tau}}, \underline{m}^{i,\hat{\tau}}, \overline{s}^{\hat{t},t''})) \end{aligned} \tag{T.5.31}$$

minus

$$\Pr(\tilde{s}^{\hat{t},t'} = \overline{s}^{\hat{t},t'})u_i(\mathbf{a}^{\hat{t}+t'}(\hat{m}, \underline{m}^{i,\hat{\tau}}, \overline{s}^{\hat{t},t'}))+ \sum_{t''=t'+1}^{\infty} \delta^{(t''-t')} \sum_{s^{\hat{t},t''} \in S_4^{t''}(\overline{s}^{\hat{t},t'})} \Pr(\tilde{s}^{\hat{t},t''} = \overline{s}^{\hat{t},t''})u_i(\mathbf{a}^{\hat{t}+t''}(\hat{m}, \underline{m}^{i,\hat{\tau}}, \overline{s}^{\hat{t},t''})) \tag{T.5.32}$$

is greater or equal to

$$\Pr(\tilde{s}^{\hat{t},t'} = \overline{s}^{\hat{t},t'}) \left\{ \frac{\delta q \left[1 - \left(\frac{\delta q}{1-\delta(1-q)}\right)^T\right](\overline{v}_i^i - \underline{\omega}_i^i)}{(1-\delta)} - (\overline{u}_i - \underline{u}_i) \right\} \tag{T.5.33}$$

Notice now that we know that the quantity in (T.5.33) is in fact positive for δ sufficiently close to 1. This is simply because the term in curly brackets in (T.5.33) is the same as the right-hand side of (T.5.18). Therefore, we have dealt with any $\overline{s}^{\hat{t},t'} \in S_3^{t'}$ and with all its relevant "successors" of the form $S_4^{t''}(\overline{s}^{\hat{t},t'})$. Since t' is arbitrary, by (T.5.27), this exhausts $S_3^{t'}$ and $S_4^{t'}$ for all possible values of t'.

Finally, we deal with $s^{\hat{t},t'} \in S_5^{t'}$. Notice first of all that if we fix any $\overline{s}^{\hat{t},t'} \in S_5^{t'}$, then it follows from (T.5.21) and (T.5.28) that any $s^{\hat{t},t'}$ of the form $[\overline{s}^{\hat{t},t'-1}, s^{t'-1,t'}]$ (where $\overline{s}^{\hat{t},t'-1}$ are the first $t'-1$ triples of $\overline{s}^{\hat{t},t'}$) is in fact in $S_5^{t'}$.

Using, (T.5.28) and Definitions T.1.1 and T.1.2 we get

$$\sum_{s^{t'-1,t'} \in S^1} \Pr(\tilde{s}^{t'-1,t'} = s^{t'-1,t'})\, u_i(\mathbf{a}^{\hat{t}+t'}(\underline{m}^{i,\hat{\tau}}, \underline{m}^{i,\hat{\tau}}, [\overline{s}^{\hat{t},t'-1}, s^{t'-1,t'}])) = q\overline{v}_i^i + (1-q)z_i > q\underline{\omega}_i^i + (1-q)z_i \geq \sum_{s^{t'-1,t'} \in S^1} \Pr(\tilde{s}^{t'-1,t'} = s^{t'-1,t'})\, u_i(\mathbf{a}^{\hat{t}+t'}(\hat{m}, \underline{m}^{i,\hat{\tau}}, [\overline{s}^{\hat{t},t'-1}, s^{t'-1,t'}])) \tag{T.5.34}$$

Therefore, since the $\overline{s}^{\hat{t},t'}$ that we fixed is an arbitrary element of $S_5^{t'}$, we can now conclude that

$$\sum_{s^{\hat{t},t'} \in S_5^{t'}} \Pr(\tilde{s}^{\hat{t},t'} = s^{\hat{t},t'})\, u_i(\mathbf{a}^{\hat{t},t'}(\underline{m}^{i,\hat{\tau}}, \underline{m}^{i,\hat{\tau}}, s^{\hat{t},t'})) \geq \sum_{s^{\hat{t},t'} \in S_5^{t'}} \Pr(\tilde{s}^{\hat{t},t'} = s^{\hat{t},t'})\, u_i(\mathbf{a}^{\hat{t},t'}(\hat{m}, \underline{m}^{i,\hat{\tau}}, s^{\hat{t},t'})) \tag{T.5.35}$$

Hence, the proof of the lemma is now complete. ■

Remark T.5.1: *Let the strategy profile (g, μ) described in Definitions T.1.1 and T.1.2 be given. Consider a player $\langle i, t\rangle$, and a realization of future uncertainty $s^{t,t'}$ as defined in the proof of Lemma T.5.4.*

Let any message $m \in M_i^{t+1}$ be given, and fix any information set $\mathcal{I}_i^{tE}$ and associated beliefs $\Phi_i^{tE}(\cdot)$.

It is then clear from Definitions T.1.1 and T.1.2 and T.5.1, that for any t' the action that player $\langle i, t\rangle$ expects player $\langle i, t+t'\rangle$ to take is uniquely determined by m, $s^{t,t'}$ and $\mathcal{I}_i^{tE}$.

For the rest of the argument we will denote this by $\mathbf{a}_i^{t+t'}(m, s^{t,t'}, \mathcal{I}_i^{tE})$.

Lemma T.5.5: *There exists a $\underline{\delta} \in (0,1)$ such that whenever $\delta > \underline{\delta}$ the message-stage strategies described in Definition T.1.2 are sequentially rational given the beliefs described in Definition T.4.2 for every player $\langle i, t\rangle$.*

Proof: Consider any information set $\mathcal{I}_i^{tE} \in \boldsymbol{\mathcal{I}}_i^{tE}(\underline{i},\tau)$, where $\boldsymbol{\mathcal{I}}_i^{tE}(\underline{i},\tau)$ is as in Definition T.5.1. It is then evident from Lemma T.5.4 and from the beliefs $\Phi_i^{tE}(\underline{i},\tau)$ described in Definition T.5.1 that for δ sufficiently close to 1, the message strategies described in Definition T.1.2 are sequentially rational at any such information set.

From now on, consider any information set $\mathcal{I}_i^{tE} \notin \boldsymbol{\mathcal{I}}_i^{tE}(\underline{i},\tau)$. Let $m \in M_i^{t+1}$ be the message that player $\langle i,t\rangle$ should send according to the strategy μ_i^t, and let $\hat{m}$ be any other message in M_i^{t+1}. Consider a particular realization $\overline{s}^{t,t'}$, and for any $t'' \in \{1,\ldots,t'-1\}$, let $\overline{s}^{t,t''}$ denote the first t'' triples of $\overline{s}^{t,t'}$.

Next, assume that $\mathbf{a}_i^{t+t'}(m,\overline{s}^{t,t'},\mathcal{I}_i^{tE}) \neq \mathbf{a}_i^{t+t'}(\hat{m},\overline{s}^{t,t'},\mathcal{I}_i^{tE})$, and that either $t'=1$, or alternatively that $\mathbf{a}_i^{t+t''}(m,\overline{s}^{t,t''},\mathcal{I}_i^{tE}) = \mathbf{a}_i^{t+t''}(\hat{m},\overline{s}^{t,t''},\mathcal{I}_i^{tE})$ for every $t'' \in \{1,\ldots,t'-1\}$.

Clearly, in periods $\{t+1,\ldots,t'-1\}$, conditional on $\overline{s}^{t,t'}$, the payoff to player $\langle i,t\rangle$ is unaffected by the deviation to $\hat{m}$. Now consider the payoff to player $\langle i,t\rangle$, conditional on $\overline{s}^{t,t'}$, from the beginning of period t' on, for simplicity discounted from the beginning of period t'. If player $\langle i,t\rangle$ sends message m as prescribed by μ_i^t, and δ is close enough to 1, the payoff in question is bounded below by

$$(1-\delta)\underline{u}_i + \delta(q\overline{v}_i^i + (1-q)z_i) \tag{T.5.36}$$

Now consider the payoff to player $\langle i,t\rangle$ if he sends message $\hat{m}$, conditional on $\overline{s}^{t,t'}$, from the beginning of period t' on, for simplicity discounted from the beginning of period t'. In period t' the action played cannot yield him more than $\overline{u}_i$. From Lemma T.5.4, we know that, for δ close enough to 1, from the beginning of period $t'+1$ the payoff is bounded above by $\ddot{v}_i(\underline{i},T)$. Hence, for δ close enough to 1, using (T.5.9) the payoff in question is bounded above by

$$\delta\overline{u}_i + (1-\delta)\left\{\left[1-\left(\frac{\delta q}{1-\delta(1-q)}\right)^T\right]\left[q\underline{\omega}_i^i + (1-q)z_i\right] + \left(\frac{\delta q}{1-\delta(1-q)}\right)^T\left[q\overline{v}_i^i + (1-q)z_i\right]\right\} \tag{T.5.37}$$

Notice now that the quantity in (T.5.36) is the same as the quantity in (T.5.15), and the quantity in (T.5.37) is in fact the same as the quantity in (T.5.16). Hence, exactly as in the proof of Lemma T.5.3, we know that, for δ sufficiently close to 1, the quantity in (T.5.36) is greater than the quantity in (T.5.37). This is clearly enough to conclude the proof. ■

T.6. Proof of Theorem A.1: Consistency of Beliefs

Remark T.6.1: *Let $(g_\varepsilon,\mu_\varepsilon)$ be the completely mixed strategy profile of Definitions A.17 and A.18. It is then straightforward to check that as $\varepsilon \to 0$ the profile $(g_\varepsilon,\mu_\varepsilon)$ converges pointwise (in fact uniformly) to the equilibrium strategy profile described in Definitions T.1.1 and T.1.2, as required.*

Lemma T.6.1: *The strategy profile (g,μ) described in Definitions T.1.1 and T.1.2 and the beginning-of-period beliefs described in Definition T.4.1 are consistent.*

Proof: When $t=0$, there is nothing to prove. Assume $t \geq 1$. We consider two cases. First assume that player $\langle i,t\rangle$ receives message $m \in \{m^*\} \cup \breve{M}_{-i} \cup \underline{M}(i,t)$. Clearly, this is on the equilibrium path generated by the profile of strategies (g,μ) described in Definitions T.1.1 and T.1.2. Therefore, consistency in this case simply requires checking that the beginning-of-period beliefs described in Definition T.4.1 are obtained via Bayes' rule from the profile (g,μ). This is a routine exercise, and we omit the details.

Now assume that player $\langle i,t\rangle$ receives message $m \notin \{m^*\} \cup \breve{M}_{-i} \cup \underline{M}(i,t)$. From Definition T.4.1 it is immediate to check that in this case player $\langle i,t\rangle$ assigns probability one to the event that $m_{-i}^t = (m,\ldots,m)$. Given (g,μ), this event may of course have been generated by several possible histories. Notice however, that the profile (g,μ) is such that a *single* deviation by one player at the action stage is sufficient to generate the message profile $m^t = (m,\ldots,m)$. Therefore, upon observing $m \notin \{m^*\} \cup \breve{M}_{-i} \cup \underline{M}(i,t)$ the probability that

$m^t_{-i} = (m, \ldots, m)$ is an infinitesimal in ε of order no higher than 2. This needs to be compared with the probability that $m^t_{-i} \neq (m, \ldots, m)$ and $m^t_i = m$. The latter event is impossible given the profile (g, μ) unless a deviation at the message stage has occurred at some point. Therefore its probability is an infinitesimal in ε of order no lower than $2n + 1$. This is obviously enough to prove the claim. ■

Lemma T.6.2: *The strategy profile (g, μ) described in Definitions T.1.1 and T.1.2 and the end-of-period beliefs described in Definition T.4.2 are consistent.*

Proof: The case $t = 0$ is trivial. Assume $t \geq 1$, and consider any player $\langle i, t\rangle$ after having observed (m^t_i, x^t, a^t, y^t).

We deal first with the case in which $x^t = x(\kappa)$ with $\kappa > \overline{\kappa}$. Let $x(\ell_{00}, \ell^*)$ denote the realization x^t. In this case, the action-stage strategies described in Definition T.1.1 prescribe that every player $\langle k \in I, t\rangle$ should play $a^t_k(\ell^*)$. Therefore, if the observed action profile a^t is equal to $a(\ell^*)$, player $\langle i, t\rangle$ does not revise his beginning-of-period beliefs during period t. Hence consistency in this case follows immediately from the profile μ and from the consistency of beginning-of-period beliefs, which of course was proved in Lemma T.6.1. Notice now that if $a^t \neq a(\ell^*)$, then the message strategies described in Definition T.1.2 prescribe that each player $\langle k \in I, t\rangle$ should send a message that does not depend on the message m^t_k he received. Hence, in this case consistency is immediate from Definition T.4.2 and the profile μ.

We now turn to the case in which $x^t = x(\kappa)$ with $\kappa \leq \overline{\kappa}$. Here, it is necessary to consider several subcases, depending on the message m received by player $\langle i, t\rangle$. Assume first that $m \notin \breve{M}_{-i} \cup \underline{M}(i, t)$. Then for any possible triple (x^t, a^t, y^t) we have that

$$\lim_{\varepsilon \to 0} \Pr(m^t_{-i} = (m, \ldots, m) \mid m^t_i = m, x^t, a^t, g_\varepsilon, \mu_\varepsilon) = 1 \tag{T.6.1}$$

To see this consider two sets of possibilities. First, $m = m^*$, $x^t = x(\cdot, \hat{\ell}, \cdots)$, and $a^t = (a_1(\hat{\ell}), \ldots, a_n(\hat{\ell}))$. Then play is as prescribed by the equilibrium path generated by the profile (g, μ), and from Definitions T.1.1 and T.1.2 there is nothing more to prove. For all other possibilities, notice that the event $m^t = (m, \ldots, m)$ is consistent with any a^t together with n deviations at the action stage of the second type described in Definition A.17. Therefore, for any a^t, the probability of $m^t = (m, \ldots, m)$ and a^t is an infinitesimal in ε of order no higher than $2n$. On the other hand, from Definition A.18 it is immediate that the probability that $m^t_{-i} \neq (m, \ldots, m)$ (since it requires at least one deviation at the message stage) is an infinitesimal in ε of order no lower than $2n + 1$. Hence (T.6.1) follows. From (T.6.1) it is a matter of routine to check the consistency of end-of-period beliefs from using the profile (g, μ). We omit the details.

Still assuming that $x^t = x(\kappa)$ with $\kappa \leq \overline{\kappa}$, now consider the case $m = \breve{m}_j \in \breve{M}_{-i}$. In this case we can show that

$$\lim_{\varepsilon \to 0} \Pr(m^t_{-i-j} = (\breve{m}_j, \ldots, \breve{m}_j) \text{ and } m^t_j \in \underline{M}(j, t) \mid m^t_i = \breve{m}_j, x^t, a^t, g_\varepsilon, \mu_\varepsilon) = 1 \tag{T.6.2}$$

using an argument completely analogous to the one we used for (T.6.1). The details are omitted. As in the previous case, from (T.6.2) it is a matter of routine to check the consistency of end-of-period beliefs from using the profile (g, μ).

The last case remaining is $x^t = x(\kappa)$ with $\kappa \leq \overline{\kappa}$ and $m = \underline{m}^{i,\tau}$. In this case we have that

$$\begin{aligned}\lim_{\varepsilon \to 0} \Pr(m^t_{-i} = (\breve{m}_i, \ldots, \breve{m}_i) \mid m^t_i = \underline{m}^{i,\tau}, x^t, a^t, g_\varepsilon, \mu_\varepsilon) +\\ \lim_{\varepsilon \to 0} \Pr(m^t_{-i} = (\underline{m}^{i,\tau}, \ldots, \underline{m}^{i,\tau}) \mid m^t_i = \underline{m}^{i,\tau}, x^t, a^t, g_\varepsilon, \mu_\varepsilon) = 1\end{aligned} \tag{T.6.3}$$

Again, the argument is completely analogous to the one used for (T.6.1) and (T.6.2), and the details are omitted. Now take (T.6.3) as given and let $x^t = x(\cdots, i_\ell, \cdots)$.

Suppose next that $a^t_{-i} = \breve{a}^i_{-i}(i_\ell)$. Then player $\langle i, t\rangle$ does not revise his beginning-of-period beliefs, and hence, using the profile μ and Lemma T.6.1 it is immediate to check that his end-of-period beliefs are consistent in this case.

Now suppose that for some $j \neq i$ we have that $a^t_j \neq \breve{a}^i_j(i_\ell)$ and $a^t_{-i-j} = \breve{a}^i_{-i-j}(i_\ell)$. Consistency of beliefs in this case requires showing that the first element in the sum in (T.6.3) is equal to 1. Of course given (T.6.3) it suffices to compare the probabilities of the two events $m^t_{-i} = (\breve{m}_i, \ldots, \breve{m}_i)$ and $m^t_{-i} = (\underline{m}^{i,\tau}, \ldots, \underline{m}^{i,\tau})$. The first is compatible with a single deviation at the action stage on the part of player $\langle j, t\rangle$. Therefore its probability is an infinitesimal in ε of order no higher than 2. The latter requires an action-stage deviation in some period $t' < t$ (order 2 in ε), and $n - 2$ action-stage deviations in period t (order 1 each). Hence, player $\langle i, t\rangle$ has consistent beliefs if he assigns probability 1 to $m^t_{-i} = (\breve{m}_i, \ldots, \breve{m}_i)$. The consistency of his end-of-period beliefs can then be checked from the profile μ.

Now suppose that for some $j \neq i$ we have that $a^t_j \neq a^i_j(i_\ell)$ and $a^t_{-i-j} = a^i_{-i-j}(i_\ell)$. Consistency of beliefs in this case requires showing that the second element in the sum in (T.6.3) is equal to 1. Of course given (T.6.3) it suffices to compare the probabilities of the two events $m^t_{-i} = (\breve{m}_i, \ldots, \breve{m}_i)$ and $m^t_{-i} = (\underline{m}^{i,\tau}, \ldots, \underline{m}^{i,\tau})$. The first requires $(n - 2)$ deviations at the action-stage of period t, each of order 2 in ε. Since $n \geq 4$, this is therefore an infinitesimal in ε of order no lower than 4. The second is consistent with a deviation of order 2 in ε at the action-stage of some period $t' < t$, together with a deviation of order 1 in ε at the action stage of period t. Therefore its probability is an infinitesimal in ε of order no higher than 3. Hence, player $\langle i, t\rangle$ has consistent beliefs if he assigns probability 1 to $m^t_{-i} = (\underline{m}^{i,\tau}, \ldots, \underline{m}^{i,\tau})$. The consistency of his end-of-period beliefs can then be checked from the profile μ. The same argument applies to show the consistency of his end-of-period beliefs when $a^t_{-i} = a^i_{-i}(i_\ell)$. We omit the details.

In all other possible cases for a^t, the messages sent by all players $\langle j \neq i, t\rangle$ do not in fact depend on a^t, provided that m^t_j is either $\breve{m}_i$ or $\underline{m}^{i,\tau}$. Given (T.6.3), the consistency of the end-of-period beliefs of player $\langle i, t\rangle$ can then be checked directly from the profile μ. ■

T.7. Proof of Theorem A.1

Given any $v^* \in \text{int}(V)$ and any $\delta \in (0, 1)$, using (A.10), (A.8) and the strategies and randomization devices described in Definitions A.7, A.8, T.1.1 and T.1.2 clearly implement the payoff vector v^*.

From Lemmas T.5.3 and T.5.5 we know that there exists a $\underline{\delta} \in (0, 1)$ such that whenever $\delta > \underline{\delta}$ each strategy in the profile described in Definitions T.1.1 and T.1.2 is sequentially rational given the beliefs described in Definitions T.4.1 and T.4.2.

From Lemmas T.6.1 and T.6.2 we know that the strategy profile described in Definitions T.1.1 and T.1.2 and the beliefs described in Definitions T.4.1 and T.4.2 are consistent.

Hence, using Lemma T.3.1, the proof of Theorem A.1 is now complete. ■