Monetary and Fiscal Policy Coordination when Bonds Provide Transactions Services –Technical Appendix–

M. Canzoneri R. Cumby B. Diba D. López-Salido*

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Abstract

This Appendix contains a detailed description of the (i) model, (ii) the steady state, (iii) and the (log-linear) approximation of the (reduced) system used in the paper "Monetary and Fiscal Policy Coordination when Bonds Provide Transactions Services".

1 The model

The economy consists of households, a continuum of firms producing differentiated intermediate goods, a perfectly competitive final goods firm; and finally, the government and the central bank will be in charge of fiscal and monetary policies, respectively. Next we describe the objectives and constraints of the different agents.

1.1 Households

Let c_t and n_t represent consumption and hours of the households, respectively. Preferences are defined by the discount factor $\beta \in (0,1)$ and the period utility $U(c_t, n_t)$. Households seek to maximize $E_0 \sum_{t=0}^{\infty} \beta^t \, U(c_t, n_t)$, subject to the sequence of budget constraints,

$$b_t + m_t \ + (1+\tau_t)c_t = w_t n_t + \pi_t^{-1} \ (R_{t-1} b_{t-1} + m_{t-1}) \ + \mathfrak{d}_t + t_t \tag{1}$$

where $b_t = \frac{B_t}{P_t}$, and $m_t = \frac{M_t}{P_t}$. At the beginning of the period the consumer receives labor income $w_t n_t$ (where w_t denotes the real wage), and income from holding riskless one-period bonds carried over t from period $t-1$. R_{t-1} denotes the gross nominal return on bonds purchased in period $t-1$. $\mathfrak{d}_t$ are dividends from ownership of firms, and t_t are real lump sum taxes, $\pi_t = \frac{p_t}{p_{t-1}}$ is the gross inflation rate . As in Smichtt-Grohé and Uribe (2004), we assume that transaction costs are proportional to consumption, and the factor of proportionality, τ_t, is related to velocity, v_t, as follows

$$\tau_t = A v_t + \frac{B}{v_t} - 2(AB)^{.5} \tag{2}$$

*Canzoneri, Cumby and Diba, Georgetown University, Department of Economics; Lopez-Salido, Federal Reserve Board. The opinions expressed here are solely those of the authors and do not necessarily reflect the views of the Board of Governors of the Federal Reserve System or of anyone else associated with the Federal Reserve System.

where $A > 0$, $B > 0$. Notice that there is a satiation level of velocity, $v^* = (B/A)^{.5}$, such that $\tau_t(v^*) = 0$. It is interesting to write the previous expression as follows

$$\tau_t = \frac{A}{v_t}[(v^*)^2 - (v_t)^2] + \vartheta_t \tag{3}$$

where

$$\vartheta_t = 2A(v_t - v^*) \tag{4}$$

Notice that, in equilibrium, $v_t \geq v^*$, i.e. $\vartheta_t \geq 0$, and $\tau_t \geq 0$. Velocity is defined as follows

$$v_t = \frac{c_t}{\widetilde{m}_t} \tag{5}$$

where $\widetilde{m}_t$ is a CES bundle

$$\widetilde{m}_t^{\rho} = a^{1-\rho}\ m_t^{\rho}\ +\ (1-a)^{1-\rho}\ b_t^{\rho} \tag{6}$$

Notice that, at the steady state, the satiation level of transactions is proportional to the level of consumption, i.e. $\widetilde{m}^* = \frac{c}{v^*}$. Notice that $\rho = 0$ corresponds to the Cobb Douglas case, and $\rho = 1$ to the case of perfect substitution between money and bonds.

The first order conditions for the optimizing consumer's problem can be written as:

$$w_t = -\frac{U_{n_t}}{\lambda_t} \tag{7}$$

$$U_{c,t} = \lambda_t \left\{1 + \vartheta_t\right\} \tag{8}$$

$$\left\{1 - A[v_t^2 - (v^*)^2]\left(\frac{a\ \widetilde{m}_t}{m_t}\right)^{1-\rho}\right\} = (R_t^*)^{-1} \tag{9}$$

$$\left\{1 - A[v_t^2 - (v^*)^2]\left(\frac{(1-a)\ \widetilde{m}_t}{b_t}\right)^{1-\rho}\right\} = \frac{R_t}{R_t^*} \tag{10}$$

where λ_t is the marginal value of real wealth, and R_t^* is the CCAPM interest rate, i.e.

$$(R_t^*)^{-1} \equiv \beta E_t\left\{\frac{P_t}{P_{t+1}}\frac{\lambda_{t+1}}{\lambda_t}\right\} = \beta E_t\left\{\frac{P_t}{P_{t+1}}\frac{U_{c,t+1}}{U_{c,t}}\frac{(1+\vartheta_t)}{(1+\vartheta_{t+1})}\right\} \tag{11}$$

From expressions (7) and (8) we have

$$\frac{w_t}{(1+\vartheta_t)} = -\frac{U_{n_t}}{U_{c,t}} \tag{12}$$

From expressions (9) and (10) it follows that

$$\frac{R_t^* - R_t}{R_t^* - 1} = \left(\frac{1-a}{a}\right)^{1-\rho}\left(\frac{m_t}{b_t}\right)^{1-\rho} \tag{13}$$

Preferences In addition, we specialize the period utility to take the form

$$u(c_t, n_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \chi_0\frac{n_t^{1+\chi}}{1+\chi} \tag{14}$$

where $\sigma \geq 0$, $\chi \geq 0$, and $\chi_0 > 0$.

1.2 Firms

We assume the existence of a continuum of monopolistically competitive firms producing differentiated intermediate goods. The latter are used as inputs by a (perfectly competitive) firm producing a single final good.

1.2.1 Final Goods Firm

The final good is produced by a representative, perfectly competitive firm with a constant returns technology:

$$y_t = \left(\int_0^1 x_t(j)^{\frac{\varepsilon-1}{\varepsilon}} \; dj \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{15}$$

where $x_t(j)$ is the quantity of intermediate good j used as an input. Profit maximization, taking as given the final goods price P_t and the prices for the intermediate goods $P_t(j)$, for all $j \in [0, 1]$, yields the following set of demand schedules,

$$x_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon} \; Y_t. \tag{16}$$

Finally, the zero profit condition yields the following price index

$$P_t = \left(\int_0^1 P_t(j)^{1-\varepsilon} \; dj \right)^{\frac{1}{1-\varepsilon}}. \tag{17}$$

1.2.2 Intermediate Goods Firm

Technology The production function for a typical intermediate goods firm (say, the one producing good j) is given by:

$$y_t(j) = a_t \; k \; n_t(j)^{\alpha} \tag{18}$$

where a_t represents total factor productivity, and k and $n_t(j)$ represent a fixed amount of capital and labor services hired by firm j, and the parameter $0 < \alpha < 1$. We assume that

$$\ln(a_t) = (1 - \rho_a) \ln a + \rho_a \; \ln(a_{t-1}) \; + \; \varepsilon_{a_t} \tag{19}$$

Given that there is *no reallocation* among capital across firms over time, and that capital is fixed at the firm level, then *aggregate capital is also fixed.* Under these circumstances, the firm's marginal cost will depend inversely on its relative price (with respect to the aggregate). Hence, the optimal decision for labor is given by the standard equation relating the real wage and the marginal product of labor, that implicitly defines the real marginal cost that, given the previous assumption, it will vary for each firm

$$w_t = mc_t(j) \; \alpha \; \left(\frac{y_t(j)}{n_t(j)} \right) \tag{20}$$

Price Setting Intermediate firms are assumed to set nominal prices in a staggered fashion, according to the stochastic time dependent rule proposed by Calvo (1983). Each firm resets its price with probability $1 - \theta$ each period, independently of the time elapsed since the last adjustment. Thus, each period a measure $1 - \theta$ of producers reset their prices, while a fraction θ of producers do not re-optimize their prices. Consider a firm which last set its price $\widetilde{P}_t$ at time t. We assume that the price accruing to that firm at time $t + j$ (before the next reoptimization) is given by $\pi^j \widetilde{P}_t$, where π is the steady state inflation rate.

Optimal Price In general, a firm resetting its price in period t will seek to maximize

$$\max_{\{\widetilde{P}_t\}} \sum_{k=0}^{\infty} \theta^k \; E_t \left\{ \Lambda_{t,t+k} \; y_{t+k}(j) \; \left(\pi^k \widetilde{P}_t - P_{t+k} \; w_{t+k} n_{t+k}(j) \right) \right\}$$

notice that, given the Cobb-Douglas specification (18), in this set-up expression (20) allows to write the total cost in terms of the real marginal costs, which implies that the previous expression can be written as follows

$$\max_{\{\widetilde{P}_t\}} \sum_{k=0}^{\infty} \theta^k \; E_t \left\{ \Lambda_{t,t+k} \; y_{t+k}(j) \; \left(\pi^k \widetilde{P}_t - P_{t+k} \; \alpha \; mc_{t+k}(j) \right) \right\} \tag{21}$$

subject to the sequence of demand constraints, $y_{t+k}(j) = \left(\frac{\pi^k \widetilde{P}_t}{P_{t+k}} \right)^{-\varepsilon} \; y_{t+k}$ where $\widetilde{P}_t$ represents the price chosen by firms resetting prices at time t, and the (nominal) stochastic discount factor, $\Lambda_{t,t+k} = \beta \frac{U_{c,t+k}}{U_{c,t}} \frac{P_t}{P_{t+k}} = \lambda_{t,t+k} \frac{P_t}{P_{t+k}}$.

We first relate the firm's marginal cost and the aggregate or average marginal cost using expressions (16) and (20)

$$mc_{t+k}(j) = \frac{w_{t+k}}{\alpha} \frac{n_{t+k}(j)}{y_{t+k}(j)} = \frac{w_{t+k}}{\alpha \; y_{t+k}/n_{t+k}} \left(\frac{\widetilde{P}_t}{P_t} \right)^{-\varepsilon\left(\frac{1-\alpha}{\alpha}\right)} \left(\frac{P_t}{P_{t+k}} \right)^{-\varepsilon\left(\frac{1-\alpha}{\alpha}\right)} \left(\pi^k \right)^{-\varepsilon\left(\frac{1-\alpha}{\alpha}\right)}$$

which implies that

$$mc_{t+k}(j) = mc_{t+k} \; \left(\frac{P_{t+k}}{P_t} \right)^{\varepsilon\left(\frac{1-\alpha}{\alpha}\right)} \; \left(\pi^k \right)^{-\varepsilon\left(\frac{1-\alpha}{\alpha}\right)} \; \left(\frac{\widetilde{P}_t}{P_t} \right)^{-\varepsilon\left(\frac{1-\alpha}{\alpha}\right)} \tag{22}$$

Notice that using the previous expression it is possible to write the original problem (21) as follows, after dividing by P_t,

$$\max_{\{\widetilde{P}_t\}} \sum_{k=0}^{\infty} \theta^k \; E_t \left\{ \Lambda_{t,t+k} \; y_{t+k}(j) \; \left(\pi^k \frac{\widetilde{P}_t}{P_t} - \frac{P_{t+k}}{P_t} \; \widetilde{mc}_{t+k} \left(\frac{\widetilde{P}_t}{P_t} \right)^{-\varepsilon\left(\frac{\alpha-1}{\alpha}\right)} \right) \right\}$$

where according to expression (22)

$$\widetilde{mc}_{t+k} = mc_{t+k} \left(\frac{P_{t+k}}{P_t} \right)^{\varepsilon\left(\frac{1-\alpha}{\alpha}\right)} \left(\pi^k \right)^{-\varepsilon\left(\frac{1-\alpha}{\alpha}\right)} \tag{23}$$

is a function only of aggregate variables. The first order condition for this problem is given by

$$\sum_{k=0}^{\infty} \theta^k \; E_t \left\{ \Lambda_{t,t+k} \; y_{t+k}(j) \; \left(\pi^k \frac{\widetilde{P}_t}{P_t} - \frac{\varepsilon}{(\varepsilon - 1)} \; \left(\frac{\widetilde{P}_t}{P_t} \right)^{-\varepsilon\left(\frac{1-\alpha}{\alpha}\right)} \frac{P_{t+k}}{P_t} \; \widetilde{mc}_{t+k} \right) \right\} = 0 \tag{24}$$

The previous expression can be written as follows

$$\sum_{k=0}^{\infty} \theta^k \; E_t \left\{ \Lambda_{t,t+k} \; y_{t+k}(j) \; \left(\pi^k \left(\frac{\widetilde{P}_t}{P_t} \right)^{1+\varepsilon\left(\frac{1-\alpha}{\alpha}\right)} - \frac{\varepsilon}{(\varepsilon - 1)} \; \frac{P_{t+k}}{P_t} \; \widetilde{mc}_{t+k} \right) \right\} = 0 \tag{25}$$

This can be used to solve for the price of newly reset prices relative to the aggregate price level as follows

$$\left(\frac{\widetilde{P}_t}{P_t}\right)^{1+\varepsilon(\frac{1-\alpha}{\alpha})} = \frac{\varepsilon}{(\varepsilon-1)} \frac{E_t \sum_{k=0}^{\infty} \theta^k \ \Lambda_{t,t+k} \ \widetilde{mc}_{t+k} \left(\frac{P_{t+k}}{P_t}\right) \ y_{t+k}(j)}{E_t \sum_{k=0}^{\infty} \theta^k \ \Lambda_{t,t+k} \ \pi^k \ y_{t+k}(j)}$$

Given the concave production function, firms that maintain a high relative price will face a lower marginal cost than the norm.[1] In the limiting case of a linear technology ($\alpha = 1$), all firms will be facing a common marginal cost.

Notice that $y_{t+k}(j) = \left(\frac{\pi^k \widetilde{P}_t}{P_{t+k}}\right)^{-\varepsilon} \ y_{t+k} = \left(\frac{\widetilde{P}_t}{P_t}\right)^{-\varepsilon} \left(\frac{P_t}{P_{t+k}}\right)^{-\varepsilon} \ \left(\pi^k\right)^{-\varepsilon} \ y_{t+k}$, hence using the previous expression we can express the relative prices as follows

$$\left(\frac{\widetilde{P}_t}{P_t}\right)^{1+\varepsilon(\frac{1-\alpha}{\alpha})} = \frac{\varepsilon}{\alpha\,(\varepsilon-1)} \frac{E_t \sum_{k=0}^{\infty} \theta^k \ \Lambda_{t,t+k} \ \widetilde{mc}_{t+k} \left(\frac{P_{t+k}}{P_t}\right)^{1+\varepsilon} \ \left(\pi^k\right)^{-\varepsilon} \ y_{t+k}}{E_t \sum_{k=0}^{\infty} \theta^k \ \Lambda_{t,t+k} \ \left(\pi^k\right)^{1-\varepsilon} \ \left(\frac{P_{t+k}}{P_t}\right)^{\varepsilon} \ y_{t+k}} \tag{26}$$

Using now expression (23) into the previous one yields

$$\left(\frac{\widetilde{P}_t}{P_t}\right)^{1+\varepsilon(\frac{1-\alpha}{\alpha})} = \frac{\varepsilon}{\varepsilon-1} \frac{E_t \sum_{k=0}^{\infty} \theta^k \ \Lambda_{t,t+k} \ mc_{t+k} \left(\frac{P_{t+k}}{P_t}\right)^{1+\frac{\varepsilon}{\alpha}} \ \left(\pi^k\right)^{-\frac{\varepsilon}{\alpha}} y_{t+k}}{E_t \sum_{k=0}^{\infty} \theta^k \ \Lambda_{t,t+k} \ \left(\pi^k\right)^{1-\varepsilon} \ \left(\frac{P_{t+k}}{P_t}\right)^{\varepsilon} \ y_{t+k}}$$

i.e. we can write the previous expression in terms of the real stochastic discount factor

$$\left(\frac{\widetilde{P}_t}{P_t}\right)^{1+\varepsilon(\frac{1-\alpha}{\alpha})} = \frac{\varepsilon}{\varepsilon-1} \frac{E_t \sum_{k=0}^{\infty} \theta^k \ \lambda_{t,t+k} \ mc_{t+k} \ \left(\frac{P_{t+k}/P_t}{\pi^k}\right)^{\frac{\varepsilon}{\alpha}} y_{t+k}}{E_t \sum_{k=0}^{\infty} \theta^k \ \lambda_{t,t+k} \ \left(\frac{P_{t+k}/P_t}{\pi^k}\right)^{\varepsilon-1} \ y_{t+k}}$$

This expression can be written as follows:

$$\left(\frac{\widetilde{P}_t}{P_t}\right)^{1+\varepsilon(\frac{1-\alpha}{\alpha})} = \frac{\varepsilon}{\varepsilon-1} \frac{z_{2,t}}{z_{1,t}} \tag{27}$$

where the variables $z_{1,t}$ and $z_{2,t}$ are defined as

$$z_{2,t} \ = \ \beta\theta \ E_t\{\left(\frac{\pi_{t+1}}{\pi}\right)^{(\frac{\varepsilon}{\alpha})} z_{2,t+1}\} \ + \ \lambda_t \ mc_t \ y_t \tag{28}$$

$$z_{1,t} \ = \ \beta\theta \ E_t\{\left(\frac{\pi_{t+1}}{\pi}\right)^{\varepsilon-1} z_{1,t+1}\} \ + \lambda_t \ y_t \tag{29}$$

Aggregate Price The equation describing the dynamics for the aggregate price level is given by

$$P_t = \left[\theta \ \left(\pi P_{t-1}\right)^{1-\varepsilon} + (1-\theta) \ \left(\widetilde{P}_t\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}. \tag{30}$$

[1] Hence, given that a fraction of firms do not change prices during any given period, producers that can adjust their prices upward suffer from a reduction in their relative demand which, in turn, implies that their marginal costs will rise less that proportionally with that of the average. This strategic complementarity effect is reflected in their price decisions so increasing the degree of price sluggishness of the economy. See Woodford (2003) and below for details.

Dividing by the aggregate price level the expression (30) yields

$$1 = \theta \left(\frac{\pi}{\pi_t}\right)^{1-\varepsilon} + (1-\theta)\left(\frac{\widetilde{P}_t}{P_t}\right)^{1-\varepsilon} \tag{31}$$

which it defines the aggregate rate of price inflation. We can use (31) to solve for inflation. This yields

$$\pi_t = \pi \left[(\theta)^{-1}\left(1-(1-\theta)\left(\widetilde{P}_t/P_t\right)^{1-\varepsilon}\right)\right]^{\varepsilon-1}$$

From the previous expression, it is interesting to note that when firms index optimal prices between adjustment to be equal to the inflation rate, the optimally set prices are equal to aggregate price level, i.e. $\frac{\widetilde{P}_t}{P_t}=1$. Finally, under flexible prices, i.e. $\theta \rightarrow 0$, then expressions (27)-(29), (31) leads to the static relationship

$$(mc_t)^{-1} = \frac{\varepsilon}{\varepsilon-1} \tag{32}$$

1.3 Government

The government's flow budget constraint can be written as follows

$$b_t + m_t = \pi_t^{-1}(R_{t-1}b_{t-1} + m_{t-1}) + g_t - t_t \tag{33}$$

1.4 Market Clearing and Aggregate Equilibrium

In a symmetric equilibrium all firms will set the same price. Hence, the firm's demand for labor is given by

$$mc_t = \alpha^{-1}\ w_t\ \frac{n_t}{y_t} \tag{34}$$

From expression (18), although there is no labor dispersion across households, there is labor dispersion across firms. Hence, we have that

$$n_t(j) = \left(\frac{y_t(j)}{a_t\ k}\right)^{1/\alpha} \tag{35}$$

where the aggregate variables are the sum of ,

$$n_t = \int_0^1 n_t(j)dj = \int_0^1 \left(\frac{y_t(j)}{a_t\ k}\right)^{1/\alpha} dj = \left(\frac{y_t}{a_t\ k}\right)^{1/\alpha} \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon/\alpha} dj$$

the second term is related to the relative price dispersion across goods. In particular, following Yun (1996), we can define $\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon/\alpha} dj = \left(\frac{P_t^*}{P_t}\right)^{-\varepsilon/\alpha} = Disp_t$, hence

$$y_t = a_t\ k\ \left(\frac{n_t}{Disp_t}\right)^{\alpha} \tag{36}$$

As in expression (30), the Calvo type of staggering implies

$$(P_t^*)^{-\varepsilon/\alpha} = \theta\ (\pi P_{t-1}^*)^{-\varepsilon/\alpha} + (1-\theta)\ \left(\widetilde{P}_t\right)^{-\varepsilon/\alpha}$$

which, in turn, can be written as

$$\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon/\alpha} = \theta \left(\frac{\pi}{\pi_t}\frac{P_{t-1}^*}{P_{t-1}}\right)^{-\varepsilon/\alpha} + (1-\theta) \left(\frac{\widetilde{P}_t}{P_t}\right)^{-\varepsilon/\alpha}$$

Hence, we obtain that the price dispersion follows

$$Disp_t = (1-\theta)\left(\frac{\widetilde{P}_t}{P_t}\right)^{-\varepsilon/\alpha} + \theta(\frac{\pi_t}{\pi})^{\varepsilon/\alpha} Disp_{t-1} \tag{37}$$

It is interesting to note that the Calvo staggering contracts imply that price dispersion today will depend upon all past relative prices. In other words, price dispersion today depends upon past price dispersion and current relative prices.

Using expression (36) yields

$$y_t = \frac{a_t\ k}{Disp_t^{\alpha}}\ {n_t}^{\alpha} \tag{38}$$

where

$$y_t = (1+\tau_t)c_t + g_t \tag{39}$$

and the government spending follows an exogenous AR(1) process

$$\ln(g_t) = (1-\rho_g)\ln g + \rho_g\ \ \ln(g_t)\ +\ \varepsilon_{g_t} \tag{40}$$

In this economy, a stationary equilibrium will be a set of processes for $\{z_{2,t},\ z_{1,t},\ Disp_t,\ mc_t,\ w_t,\ R_t^*,\ \frac{\widetilde{P}_t}{P_t},\ \pi_t,\ y_t,\ c_t,\ n_t,\ \widetilde{m}_t,\ m_t,\ b_t,\ \tau_t,\ v_t\}_{t=0}^{\infty}$, satisfying equations (2)-(6), (10)-(13), (27)-(29), (31)-(34), (37)-(39), given the initial conditions m_{-1}, b_{-1}, $Disp_{-1}$, as well as the exogenous stochastic processes for $\{a_t\}_{t=0}^{\infty}$ and $\{g_t\}_{t=0}^{\infty}$, i.e. (19) and (40). And finally, we will need two rules for the nominal interest rates and taxes t_t $\{R_t,\ t_t\}$.

2 Steady State

We will set the values of some of the parameters as to match the historical values of the ratios of bonds and money to consumption ($\frac{b}{c}$, $\frac{m}{c}$), the average inflation rate (π), and the average nominal interest rate (R). We also set the ratio of transaction costs to consumption, τ, to be 0.1 percent; and we set alternative values for the parameter ρ which determines the elasticity of substitution between money and bonds. From these values, we can obtain the steady state values for other variables and the remaining parameters of transaction costs (i.e. A, B, and v^* of expression (2)) and the parameter a of the CES expression (6).

Under full indexation of price contracts to steady state inflation there is no steady state relative price distortion, i.e. $\frac{\widetilde{P}}{P} = Disp = 1$, and hence from expression (11) we pin down the steady state value of $R^* = \frac{\pi}{\beta}$, which given R, pins down the steady state spread between the CCAPM interest rates and the interest rate on government bonds. Simple manipulations of expression (13) yields to a value for the parameter a. From the CES aggregator (6) we obtain the steady value of velocity, v, in expression (5). From expression (2) it follows that, $\tau v = A[v-v^*]^2$, which implies that the term $A[v^2-(v^*)^2] = \tau v\frac{v+v^*}{v-v^*}$, in expression (9). Then we obtain the following expression for v^*,

$$v^* = \frac{\left(1-\frac{1}{R^*}\right)v\ -\ \tau v^2\left(\frac{a\,\widetilde{m}}{m}\right)^{1-\rho}}{\left(1-\frac{1}{R^*}\right)\ +\ \tau v\left(\frac{a\,\widetilde{m}}{m}\right)^{1-\rho}}$$

Using this value for v^* and v, we obtain the parameter A from the condition (9).

From expression (4), (12), (32), (34), (38), and the (39) we can obtain the steady state value of n, for a given ratio of government spending over GDP $-\gamma_g-$

$$n^{\chi+1} = \frac{\alpha\epsilon(1+\tau)}{2A(\epsilon-1)(1-\gamma_g)(v-v^*)}$$

Output follows from (38), which given γ_g pins down government spending, then consumption follows from the market clearing condition (39). Given the steady state value of consumption we can pin down the level of bonds and money. And from the government budget constraint, (33), we obtain taxes and deficit. From the labor supply expression (12) we obtain the steady value of the real wages.

3 Equilibrium Dynamics

Consumers The log-linearized versions of expressions (2)-(6) are

$$\widehat{\tau}_t = \gamma_\tau \widehat{v}_t \tag{41}$$

$$\widehat{\vartheta}_t = \gamma_\vartheta \widehat{v}_t \tag{42}$$

$$\widehat{v}_t = \widehat{c}_t - \left[\gamma_a(\widehat{m}_t - \widehat{b}_t) + \widehat{b}_t\right] \tag{43}$$

where $\gamma_\tau \equiv \frac{v+v^*}{v-v^*}$, $\gamma_\vartheta \equiv \frac{v}{v-v^*}$, and $\gamma_a \equiv a^{1-\rho}(\frac{m}{\widetilde{m}})^\rho$. The log-linearized versions of the households' optimality conditions (10)-(13) are

$$\widehat{R}_t - \widehat{R}^*_t = -\delta_v \widehat{v}_t - (1-\rho)\gamma_a\gamma_R\ (\widehat{m}_t - \widehat{b}_t) \tag{44}$$

$$\widehat{R}^*_t - E_t\widehat{\pi}_{t+1} = -\sigma(\widehat{c}_t - E_t\widehat{c}_{t+1}) + \frac{2Av}{1+\vartheta}\ E_t\left(\widehat{v}_{t+1} - \widehat{v}_t\right) \tag{45}$$

$$\widehat{w}_t = \chi\widehat{n}_t + \sigma\widehat{c}_t + \frac{2Av}{1+\vartheta}\ \widehat{v}_t \tag{46}$$

$$\left(\frac{R^*(R-1)}{R^*-1}\right)\widehat{R}^*_t - R\ \widehat{R}_t = R(1-\rho)\gamma_R(\widehat{m}_t - \widehat{b}_t) \tag{47}$$

where $\gamma_R = \frac{R^*-R}{R}$, $\delta_v = \frac{2v^2}{v^2-(v^*)^2}\gamma_R$, $\vartheta = 2A(v-v^*)$.

Firms The log-linearization of the supply side equations (27)-(29), (31)-(34), (37)-(39) yields

$$\widehat{\pi}_t = \beta E_t\widehat{\pi}_{t+1} + \lambda_p\ \widehat{mc}_t \tag{48}$$

$$\widehat{mc}_t = \widehat{w}_t - \widehat{y}_t + \widehat{n}_t \tag{49}$$

$$\widehat{y}_t = \alpha\ \widehat{n}_t + \widehat{a}_t \tag{50}$$

$$\widehat{y}_t = \tau\gamma_c\widehat{\tau}_t + (1+\tau)\gamma_c\ \widehat{c}_t + \gamma_g\ \widehat{g}_t \tag{51}$$

where $\gamma_c = \frac{c}{y}$, $\gamma_g = \frac{g}{y}$, and the slope coefficient, $\lambda_p \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\ [1+\varepsilon(\frac{1-\alpha}{\alpha})]}$. That is, under $\alpha = 1$ this corresponds to the standard model.

Baseline Fiscal and Monetary Policy Rules We close the model with a monetary and a fiscal rule, i.e. two equations for $\widehat{R}_t$ and $\widetilde{t}_t$, respectively. Hence, we assume that the fiscal policy is governed by a rule for setting taxes as follows

$$\widetilde{t}_t = \phi\, \widehat{b}_{t-1} \tag{52}$$

where $\widetilde{t}_t = \frac{t_t - t}{b}$, represents a linear approximation of taxes relative to steady state debt.

We assume that monetary policy is governed by a rule for setting the interest rate as follows

$$\widehat{R}_t = \phi_\pi \widehat{\pi}_t \tag{53}$$

Government Budget Constraints Using expressions (52) and (53) into a log-linear approximation to expression (33) yields

$$\widehat{b}_t + \left(\frac{m}{b}\right)\widehat{m}_t + \left(\frac{R}{\pi} + \frac{m}{\pi b}\right)\widehat{\pi}_t = \left(\frac{R}{\pi} - \phi\right)\widehat{b}_{t-1} + \left(\frac{m}{\pi b}\right)\widehat{m}_{t-1} - \left(\frac{\phi_\pi R}{\pi}\right)\widehat{\pi}_{t-1} + \left(\frac{g}{b}\right)\widehat{g}_t \tag{54}$$

Equilibrium Hence, we can solve for the vector of thirteen variables $\{\widehat{\tau}_t, \widehat{\vartheta}_t, \widehat{v}_t, \widehat{c}_t, \widehat{y}_t, \widehat{n}_t, \widehat{mc}_t, \widehat{w}_t, \widehat{\pi}_t, \widehat{R}_t, \widehat{R}^*_t, \widehat{b}_t, \widehat{m}_t\}$ using equations (41)-(47), (48)-(51), (53), and (54).

System Reduction The purpose of this section is to go from the previous large system to a three variable system. We notice that from equation (47) it follows

$$\widehat{m}_t = \widehat{b}_t + \frac{1}{R(1-\rho)\gamma_R}\left(\left(\frac{R^*(R-1)}{R^*-1}\right)\widehat{R}^*_t - R\widehat{R}_t\right) \tag{55a}$$

From the previous expression and (44) it can be obtained, after simple manipulations, the following relationship

$$\widehat{R}^*_t = \frac{1-\gamma_a}{1-\psi_{R^*}}\ \widehat{R}_t\ -\ \frac{\delta_v}{1-\psi_{R^*}}\ \widehat{v}_t \tag{56}$$

where $\psi_{R^*} \equiv \frac{\gamma_a R^*(R-1)}{R(R^*-1)}$. Using equations (43) and (55a) we obtain

$$\widehat{v}_t = \widehat{c}_t - \widehat{b}_t + \frac{\gamma_a}{(1-\rho)\gamma_R}\widehat{R}_t - \frac{\psi_{R^*}}{(1-\rho)\gamma_R}\widehat{R}^*_t \tag{57}$$

Substituting (57) into the previous expression (56) yields, after simple manipulations, the following expression

$$\widehat{R}^*_t = \psi_1(\widehat{c}_t - \widehat{b}_t) + \psi_2\widehat{R}_t \tag{58}$$

where $\psi_1 \equiv \frac{\delta_v(1-\rho)\gamma_R}{(1+\psi_{R^*})(1-\rho)\gamma_R + \delta_v\psi_{R^*}}$ and $\psi_2 \equiv \frac{\delta_v\gamma_a + (1-\gamma_a)(1-\rho)\gamma_R}{(1+\psi_{R^*})(1-\rho)\gamma_R + \delta_v\psi_{R^*}}$. By using now (58) into the previous expression (57) we can express the velocity as a function of consumption, debt, and interest rates as follows

$$\widehat{v}_t = \psi_3(\widehat{c}_t - \widehat{b}_t) + \psi_4\widehat{R}_t \tag{59}$$

where $\psi_3 = \left(\frac{(1+\psi_{R^*})(1-\rho)\gamma_R}{(1+\psi_{R^*})(1-\rho)\gamma_R + \delta_v\psi_{R^*}}\right)$ and $\psi_4 = \frac{1}{(1-\rho)\gamma_R}(\gamma_a - \psi_2\psi_{R^*})$. We can use expressions (58) and (59) into expression (45) to obtain

$$\left(\psi_1 + \frac{2Av}{1+\vartheta}\psi_3 + \sigma\right)\widehat{c}_t - (\psi_1 + \frac{2Av}{1+\vartheta}\psi_3)\widehat{b}_t + \left(\psi_2 + \frac{2Av}{1+\vartheta}\psi_4\right)\widehat{R}_t = \left(\frac{2Av}{1+\vartheta}\psi_3 + \sigma\right)E_t\widehat{c}_{t+1} - \frac{2Av}{1+\vartheta}\psi_3 E_t\widehat{b}_{t+1} + \psi_4\frac{2Av}{1+\vartheta}E_t\widehat{R}_{t+1} + E_t\widehat{\pi}_{t+1}$$

Using expression (53) we obtain the following expression

$$\left(\psi_1+\frac{2Av}{1+\vartheta}\psi_3+\sigma\right)\widehat{c}_t-(\psi_1+\frac{2Av}{1+\vartheta}\psi_3)\widehat{b}_t+\left(\psi_2+\frac{2Av}{1+\vartheta}\psi_4\right)\phi_\pi\widehat{\pi}_t= \tag{60}$$
$$\left(\frac{2Av}{1+\vartheta}\psi_3+\sigma\right)E_t\widehat{c}_{t+1}-\frac{2Av}{1+\vartheta}\psi_3E_t\widehat{b}_{t+1}+\left(\psi_4\frac{2Av\phi_\pi}{1+\vartheta}+1\right)E_t\widehat{\pi}_{t+1}$$

We use expressions (41), (46), (48), and (51) to obtain

$$\widehat{\pi}_t=\beta\ E_t\{\widehat{\pi}_{t+1}\}+\lambda_p\theta_c\ \widehat{c}_t+\lambda_p\theta_v\ \widehat{v}_t+\lambda_p\theta_g\ \widehat{g}_t-\lambda_p\theta_a\ \widehat{a}_t \tag{61}$$

where $\theta_c\equiv\left(\sigma+\left(\frac{1+\chi-\alpha}{\alpha}\right)(1+\tau)\gamma_c\right)$, $\theta_v\equiv\left(\frac{2Av}{1+\vartheta}+\left(\frac{1+\chi-\alpha}{\alpha}\right)\tau\gamma_\tau\gamma_c\right)$, $\theta_g\equiv\left(\frac{1+\chi-\alpha}{\alpha}\right)\gamma_g$, and $\theta_a\equiv\left(\frac{1+\chi}{\alpha}\right)$.

We substitute into the previous expression (61) the previous expressions (53) and (59) to get

$$(1-\lambda_p\theta_v\psi_4\phi_\pi)\widehat{\pi}_t-\lambda_p(\theta_c+\psi_3\theta_v)\widehat{c}_t+\lambda_p\theta_v\psi_3\widehat{b}_t-\lambda_p\theta_g\widehat{g}_t+\lambda_p\theta_a\widehat{a}_t=\beta E_t\{\widehat{\pi}_{t+1}\} \tag{62}$$

Notice that using expressions (55a) and (58) we obtain that

$$\widehat{m}_t=\left(1-\frac{\psi_{R^*}\psi_1}{(1-\rho)\gamma_R\gamma_a}\right)\widehat{b}_t+\left(\frac{\psi_{R^*}\psi_1}{(1-\rho)\gamma_R\gamma_a}\right)\widehat{c}_t+\left(\phi_\pi\frac{\psi_{R^*}\psi_2-\gamma_a}{(1-\rho)\gamma_R\gamma_a}\right)\widehat{\pi}_t$$

We use this expression to eliminate $\widehat{m}_t$ and $\widehat{m}_{t-1}$ from expression (54). This yields

$$\widehat{z}_t=\left(\frac{R}{\pi}-\phi+\frac{m}{b\pi}+\frac{m}{b\pi}\frac{\psi_{R^*}\psi_1}{(1-\rho)\gamma_R\gamma_a}\right)\widehat{b}_{t-1}+\frac{m}{b\pi}\left(\frac{\psi_{R^*}\psi_2}{(1-\rho)\gamma_R\gamma_a}\right)\widehat{c}_{t-1}+$$
$$\phi_\pi\left(\frac{m}{b\pi}\left(\frac{\psi_{R^*}\psi_2-\gamma_a}{(1-\rho)\gamma_R\gamma_a}\right)+\frac{R}{\pi}\right)\widehat{\pi}_{t-1}+\left(\frac{g}{b}\right)\widehat{g}_t \tag{63}$$

where

$$\widehat{z}_t\equiv\left(1+\frac{m}{b}-\frac{m}{b}\frac{\psi_{R^*}\psi_1}{(1-\rho)\gamma_R\gamma_a}\right)\widehat{b}_t+\frac{m}{b}\left(\frac{\psi_{R^*}\psi_1}{(1-\rho)\gamma_R\gamma_a}\right)\widehat{c}_t+\left(\phi_\pi\frac{m}{b}\frac{\psi_{R^*}\psi_2-\gamma_a}{(1-\rho)\gamma_R\gamma_a}+\frac{R}{\pi}+\frac{m}{b\pi}\right)\widehat{\pi}_t \tag{64}$$

Hence, from the government budget constraint it follows that there exists a linear combination of the variables $\widehat{c}_t$, $\widehat{b}_t$, and $\widehat{\pi}_t$, say the variable $\widehat{z}_t$, which is predetermined (expression (63)). Hence, we use the definition (64) to express $\widehat{b}_t$ as follows

$$\widehat{b}_t=\psi_z\widehat{z}_t+\psi_c\widehat{c}_t+\psi_\pi\widehat{\pi}_t \tag{65}$$

where ψ_z=$\left(1+\frac{m}{b}-\frac{m}{b}\frac{\psi_{R^*}\psi_1}{(1-\rho)\gamma_R\gamma_a}\right)^{-1}$, ψ_c=-$\psi_z\frac{m}{b}\left(\frac{\psi_{R^*}\psi_1}{(1-\rho)\gamma_R\gamma_a}\right)$, and ψ_π=$\psi_z\left(\phi_\pi\frac{m}{b}\frac{\psi_{R^*}\psi_2-\gamma_a}{(1-\rho)\gamma_R\gamma_a}+\frac{R}{\pi}+\frac{m}{b\pi}\right)$. We will use this expression to eliminate $\widehat{b}_t$ from expressions (60), (62), and one period ahead of expression (63). This leads to the following three equations

$$\left(\frac{2Av}{1+\vartheta}\psi_3(1-\psi_c)+\sigma\right)E_t\widehat{c}_{t+1}+\left(1+\frac{2Av}{1+\vartheta}(\phi_\pi\psi_4-\psi_3\psi_\pi)\right)E_t\widehat{\pi}_{t+1}-\frac{2Av}{1+\vartheta}\psi_3\psi_z\ \widehat{z}_{t+1}=$$
$$\left(\sigma+\left(\psi_1+\frac{2Av}{1+\vartheta}\psi_3\right)(1-\psi_c)\right)\widehat{c}_t+\left(\psi_2\phi_\pi-\psi_1\psi_\pi+\frac{2Av}{1+\vartheta}(\psi_4\phi_\pi-\psi_3\psi_\pi)\right)\widehat{\pi}_t+ \tag{66}$$
$$-\left(\psi_1+\frac{2Av}{1+\vartheta}\right)\psi_z\widehat{z}_t$$

$$\beta E_t\{\widehat{\pi}_{t+1}\} = -\lambda_p(\theta_c + \psi_3\theta_v(1-\psi_c))\widehat{c}_t + (1+\lambda_p\theta_v(\psi_3\psi_\pi - \phi_\pi\psi_4))\,\widehat{\pi}_t + \lambda_p\theta_v\psi_3\psi_z\widehat{z}_t$$
$$-\lambda_p\theta_g\widehat{g}_t + \lambda_p\theta_a\widehat{a}_t \tag{67}$$

and we move one period forward expression (63) and then we substitute out $\widehat{b}_t$ using expression (65)

$$\widehat{z}_{t+1} = \left(\frac{R-1}{\pi} - \phi\right)\psi_c\widehat{c}_t + \left(\left(\frac{R-1}{\pi} - \phi\right)\psi_\pi + \phi_\pi\frac{R}{\pi} - \frac{1}{\pi}(\frac{R}{\pi} + \frac{m}{b\pi})\right)\widehat{\pi}_t +$$
$$\left(\frac{1}{\pi} + \left(\frac{R-1}{\pi} - \phi\right)\psi_z\right)\widehat{z}_t + (\frac{g}{b})\widehat{g}_{t+1} \tag{68}$$

The system has been reduced to three equations (66), (67), and (68). In a matrix form it can be written as follows,

$$\mathbf{A}E_t\{\mathbf{x}_{t+1}\} = \mathbf{B}\mathbf{x}_t \tag{69}$$

where $\mathbf{x}_t \equiv (\widehat{c}_t, \widehat{\pi}_t, \widehat{z}_t)'$. Notice that $\mathbf{x}_t = 0$ for all t, which corresponds to the perfect foresight steady state, always constitutes a solution to the above system in an non-stochastic setup. We aim to study the conditions under which the solution to (69) is unique and converges to the steady state, for any given initial value of z (i.e. an initial value for government bonds, consumption, and inflation.) In doing so we restrict our analysis to solutions of (69) (i.e., equilibrium paths) which remain within a small neighborhood of the steady state.

The elements of matrices $\mathbf{A}$ and $\mathbf{B}$ are all functions of the underlying structural parameters,

$$\mathbf{A}=\begin{pmatrix} \frac{2Av}{1+\vartheta}\psi_3(1-\psi_c)+\sigma & 1+\frac{2Av}{1+\vartheta}(\phi_\pi\psi_4 - \psi_3\psi_\pi) & -\frac{2Av}{1+\vartheta}\psi_3\psi_z \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{B}=\begin{pmatrix} \sigma + \left(\psi_1 + \frac{2Av}{1+\vartheta}\psi_3\right)(1-\psi_c) & \begin{matrix}\psi_2\phi_\pi - \psi_1\psi_\pi + \\ \frac{2Av}{1+\vartheta}(\psi_4\phi_\pi - \psi_3\psi_\pi)\end{matrix} & -\left(\psi_1 + \frac{2Av}{1+\vartheta}\right)\psi_z \\ -\lambda_p(\theta_c + \psi_3\theta_v(1-\psi_c)) & 1+\lambda_p\theta_v(\psi_3\psi_\pi - \phi_\pi\psi_4) & \lambda_p\theta_v\psi_3\psi_z \\ \left(\frac{R-1}{\pi} - \phi\right)\psi_c & \begin{matrix}\left(\frac{R-1}{\pi} - \phi\right)\psi_\pi + \phi_\pi\frac{R}{\pi} \\ -\frac{1}{\pi}(\frac{R}{\pi} + \frac{m}{b\pi})\end{matrix} & \frac{1}{\pi} + \left(\frac{R-1}{\pi} - \phi\right)\psi_z \end{pmatrix}$$

The vector $\mathbf{x}_t$ contains two non-predetermined variables (consumption and inflation) and one predetermined (z, i.e. the combination of the stock of bonds, past inflation and consumption.) Interestingly, $\mid \mathbf{A} \mid = \beta\left(\frac{2Av}{1+\vartheta}\psi_3(1-\psi_c)+\sigma\right) \neq 0$, unless $\sigma = \frac{2Av}{1+\vartheta}\psi_3(1-\psi_c)$, so the matrix $\mathbf{A}$ can, in general be inverted. Hence, the solution to (69) is unique if and only if two eigenvalues of matrix $\mathbf{A}^{-1}\mathbf{B}$ lie outside the unit circle, and one lies inside.[2] Alternatively, if there is more than one eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ inside the unit circle the equilibrium is locally indeterminate: for any initial stock of liabilities there exists a continuum of deterministic equilibrium paths converging to the steady state, and the possibility of stationary sunspot fluctuations arises. On the other hand, if all the eigenvalues $\mathbf{A}^{-1}\mathbf{B}$ lie outside the unit circle, there is no solution to (69) that converges to the steady state.

[2] See, e.g., Blanchard and Kahn (1980).

References not Included in the Main Text

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