Derivations to Accompany:

RECENT DEVELOPMENTS IN THE MACROECONOMIC STABILIZATION LITERATURE

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These notes provide some of the algebraic derivations that were left out of the published paper.

Household (h,f)'s Utility in period t –

(1) $U_t(h,f) = E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t}[u(C_\tau(h,f)) - g(N_{s,\tau}(h,f)) + v(M_\tau(h,f)/P_\tau)]$

where $C_\tau(h,f)$ is household consumption of the composite consumption good;

when we want specific functional forms, we will specify "constant elasticity" utility:

$u(C) \equiv (1-\gamma)^{-1}C^{1-\gamma}$, $g(N) \equiv A(1+\chi)^{-1}N^{1+\chi}$ and $v(M/P) = V(1-\nu)^{-1}(M/P)^{1-\nu}$.

Bundlers –

Bundler of the composite consumption good C –

(2) $C_\tau = \Pi_{s=1}^{S}C_{s,\tau}{}^{1/S}$	Consumption Good C_τ
$P_\tau = S\Pi_{s=1}^{S}P_{s,\tau}{}^{1/S}$	Price (of bundler) for the Consumption Good
$(1/S)P_\tau C_\tau = P_{s,\tau}C_{s,\tau}$	Demand (of bundler) for the Sectoral Goods

Bundler of Sectoral good $Y_{s,\tau}$ –

(3) $Y_{s,\tau} \equiv [\int_s^{s+1}Y_{s,\tau}(f)^{(\theta-1)/\theta}df]^{\theta/(\theta-1)}$, $\theta > 1$	Sectoral Good $Y_{s,\tau}$
$P_{s,\tau} = [\int_s^{s+1}P_{s,\tau}(f)^{1-\theta}df]^{1/(1-\theta)}$	Price (of bundler) for the sectoral good $Y_{s,\tau}$
$Y^d_{s,\tau}(f) = (P_{s,\tau}/P_{s,\tau}(f))^{\theta}Y_{s,\tau}$	Demand (of the bundler) for good of firm f

Bundler of Firm f's labor input –

(4) $N_{s,\tau}(f) \equiv [\int_0^1 N_{s,\tau}(h,f)^{(\phi-1)/\phi}dh]^{\phi/(\phi-1)}$, $\phi > 1$	Labor input for $Y_{s,\tau}(f)$
$W_{s,\tau}(f) = [\int_0^1 W_{s,\tau}(h,f)^{1-\phi}dh]^{1/(1-\phi)}$	Price (of bundler) for the labor input for $Y_{s,\tau}(f)$
$N^d_{s,\tau}(h,f) = (W_{s,\tau}(f)/W_{s,\tau}(h,f))^{\phi}N_{s,\tau}(f)$	Demand (of the bundler) for labor of household (h,f)

The algebra of competitive bundlers:

Here, we assume Chari, Kehoe and McGrattan's (2000) "bundlers":

The "bundler" for sector s is a competitive (or zero profit) agent who buys the firms' $Y_{s,t}(f)$ at the price $P_{s,t}(f)$, bundles them into the sectoral good $Y_{s,t} = [\int_0^1 Y_{s,t}(f)^{(\theta-1)/\theta} df]^{\theta/(\theta-1)}$; and sells it at the price $P_{s,t}$.

The bundler minimizes the cost of producing a given amount of $Y_{s,t}$:

$\min_{Y_{s,t}(f)} \int_0^1 P_{s,t}(f)Y_{s,t}(f)df$ s.t. $\bar{Y}_{s,t} = [\int_0^1 Y_{s,t}(f)^{(\theta-1)/\theta} df]^{\theta/(\theta-1)}$

$\mathcal{L} = \int_0^1 P_{s,t}(f)Y_{s,t}(f)df + \mu\{\bar{Y}_{s,t} - [\int_0^1 Y_{s,t}(f)^{(\theta-1)/\theta} df]^{\theta/(\theta-1)}\}$

Note: Lagrangian multiplier $\mu = MC = P_{s,t}$, since bundler is competitive

First Order Condition –

$P_{s,t}(f) = P_{s,t}[\int_0^1 Y_{s,t}(f)^{(\theta-1)/\theta} df]^{[\theta/(\theta-1)]-1} Y_{s,t}(f)^{[(\theta-1)/\theta]-1} = P_{s,t}Y_{s,t}^{1/\theta}Y_{s,t}(f)^{-1/\theta}$

$= P_{s,t}(Y_{s,t}(f)/Y_{s,t})^{-1/\theta} \Rightarrow Y_{s,t}(f) = (P_{s,t}(f)/P_{s,t})^{-\theta}Y_{s,t}$

To find $P_{s,t}$, use FOC to eliminate $Y_{s,t}(f)$ in $Y_{s,t} = [\int_0^1 Y_{s,t}(f)^{(\theta-1)/\theta} df]^{\theta/(\theta-1)}$

$Y_{s,t} = [\int_0^1 Y_{s,t}(f)^{(\theta-1)/\theta} df]^{\theta/(\theta-1)} = [\int_0^1 [(P_{s,t}(f)/P_{s,t})^{-\theta}Y_{s,t}]^{(\theta-1)/\theta} df]^{\theta/(\theta-1)}$

$= (Y_{s,t}P_{s,t}^{\theta})[\int_0^1 [P_{s,t}(f)^{-(\theta-1)} df]^{\theta/(\theta-1)}$

$P_{s,t}^{-\theta} = [\int_0^1 [P_{s,t}(f)^{-(\theta-1)} df]^{\theta/(\theta-1)} \Rightarrow P_{s,t} = [\int_0^1 P_{s,t}(f)^{1-\theta} df]^{1/(1-\theta)}$

Collecting results:

$P_{s,t} = [\int_0^1 P_{s,t}(f)^{1-\theta} df]^{1/(1-\theta)}$	Price of sectoral good $Y_{s,t}$
$Y^d_{s,t}(f) = (P_{s,t}/P_{s,t}(f))^{\theta}Y_s$	Demand for good output of firm f

Budget Constraint of Household (h,f) –

(5) $M_\tau(h,f) + E_\tau[\delta_{\tau,\tau+1}B_{\tau+1}(h,f)] + P_\tau C_\tau(h,f) + P_\tau T_\tau = W_{s,\tau}(h,f)N_{s,\tau}(h,f) + M_{\tau-1}(h,f) + B_\tau(h,f) + \Omega_\tau(h,f)$

where $B_{\tau+1}$ is a portfolio of state-contingent claims, $\delta_{\tau,\tau+1}$ is the stochastic discount factor, E is an expectation over states; Ω_τ are dividends, and T_τ is a lump-sum tax (or transfer).

Household-(h,f)'s Intertemporal Maximization Problem –

Max (1), s.t. the Budget Constraint, (5), and labor demand, (4):

$$\mathcal{L} = E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t}\{[u(C_\tau(h,f)) - g(N_{s,\tau}(h,f)) + v(M_\tau(h,f)/P_\tau)]$$
$$+ \lambda_\tau[W_{s,\tau}(h,f)N_{s,\tau}(h,f) + M_{\tau-1}(h,f) + B_\tau(h,f) + \Omega_\tau(h,f) - M_\tau(h,f) - E_\tau[\delta_{\tau,\tau+1}B_{\tau+1}(h,f)] - P_\tau C_\tau(h,f) - P_\tau T_\tau]\}$$

FOC that hold independent of the type of nominal inertia –

(7) C_t: $\lambda_t(h,f)P_t = u'(\cdot)$ (or with log utility) $\lambda_t(h,f)P_t = 1/C_t(h,f)$

(6) B_{t+1}: $\delta_{t,t+1} = \beta\lambda_{t+1}(h,f)/\lambda_t(h,f)$

M_t: $v'(\cdot)/P_t = \lambda_t(h,f) - \beta E_t\lambda_{t+1}(h,f) = \lambda_t(h,f)\{1 - \beta E_t[\lambda_{t+1}(h,f)/\lambda_t(h,f)]\} = \lambda_t(h,f)[1 - E_t(\delta_{t,t+1})]$

(8) M_t: $v'(\cdot) = u'(\cdot)[1 - E_t(\delta_{t,t+1})]$ (or with log utility) $M_t(h,f) = V_tP_tC_t(h,f)/[1 - E_t(\delta_{t,t+1})]$

Remarks (on complete contingent claims modeling):

1. Cochrane (2001, Ch. 3) provides a good (and brief) discussion of complete contingent claims markets. Our parsimonious notation follows Woodford (1997).
2. Following Cochrane, $p(\text{portfolio}) = \sum_\sigma pc(\sigma)B(\sigma) = \sum_\sigma \pi(\sigma)[pc(\sigma)/\pi(\sigma)]B(\sigma) = \sum_\sigma \pi(\sigma)\delta(\sigma)B(\sigma) = E[\delta(\sigma)B(\sigma)]$, where σ is the state of nature, pc(σ) is the period t price of a dollar in state σ in period t+1, B(σ) is the number of claims in the portfolio, and p(portfolio) is the price of the portfolio. The "stochastic discount factor" δ(σ) is the price of a dollar in state σ divided by the probability of state σ occurring. $B_{t+1}(h,f)$ is B(σ). (6), like (7), when forwarded, has to hold for each state. Note: all households face the same δ(σ). So, (6) & (7) imply that all households have the same *actual* rate of growth of nominal income.
3. The "risk free" rate of return –

   Consider a bond that costs 1 dollar in t and pays I dollars in all states in t+1.

   So, $1 = E_t[\delta_{t,t+1}I_t] \Rightarrow I_t^{-1} = E_t[\delta_{t,t+1}] = \beta E_t[\lambda_{t+1}/\lambda_t] = \beta E_t[P_tC_t(h,f)/P_{t+1}C_{t+1}(h,f)]$ for all (h,f)

Using labor demand curve (4), the Household Intertemporal Maximization Problem becomes:

$$\text{Max } \mathcal{L} = E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t}\{[u(C_\tau(h,f)) - g((W_{s,\tau}(h,f)/W_{s,\tau}(f))^{-\phi}N_{s,\tau}(f)) + v(M_\tau(h,f)/P_\tau)]$$
$$+ \lambda_\tau[W_{s,\tau}(h,f)[(W_{s,\tau}(h,f)/W_{s,\tau}(f))^{-\phi}N_{s,\tau}(f)] + M_{\tau-1}(h,f) + B_\tau(h,f) + \Omega_\tau(h,f) - [\text{as above}]\}$$

If (h,f) works at a flexible-wage firm:

$W_{s,t}(h,f)$: $\phi g'(\cdot)W_{s,t}(h,f)^{-1-\phi}(1/W_{s,t}(f))^{-\phi}N_{s,t}(f) + \lambda_t(1-\phi)W_{s,t}(h,f)^{-\phi}(1/W_{s,t}(f))^{-\phi}N_{s,t}(f) = 0$

$\phi g'(\cdot)W_{s,t}(h,f)^{-1-\phi}(1/W_{s,t}(f))^{-\phi}N_{s,t}(f) = (\phi-1)[u'(\cdot)/P_t]W_{s,t}(h,f)^{-\phi}(1/W_{s,t}(f))^{-\phi}N_{s,t}(f)$

$\mu_w g'(\cdot)N_{s,t}(h,f)W_{s,t}(h,f)^{-1} = [u'(\cdot)/P_t]N_{s,t}(h,f)$ where $\mu_w \equiv \phi/(\phi-1) > 1$

canceling the $N_{s,t}(f)$ (which can't be done in the sticky wage case)

$(9)_{flex}$ $W_{s,t}(h,f)/P_t = \mu_w[g'(\cdot)/u'(\cdot)] = \mu_w A_t N_{s,t}(h,f)^\chi C_t(h,f)$

Remarks:

1. The interpretation of $(9)_{flex}$:

   Can be written as: $(9)'$ $g'(\cdot) = (1/\mu_w)[W(h,f)/P]u'(\cdot)$

   The LHS of (9′) is the disutility of working one more "hour".

   If the household works one more hour, how much does it's wage bill increase?

   Recall that: $N^d(h,f) = (W(f)/W(h,f))^\phi N(f)$, so household's wage rate falls as N(h,f) rises!

   $d(W(h,f)N(h,f))/dN(h,f) = [d(W(h,f)N(h,f))/dW(h,f)]\cdot[dW(h,f)/dN(h,f)]$

   $= [d(W(h,f)N(h,f))/dW(h,f)]/[dN(h,f)/dW(h,f)]$

   $= [(1-\phi)W(h,f)^{-\phi}W(f)^\phi N(f)]/[-\phi W(h,f)^{-1-\phi}W(f)^\phi N(f)] = [(\phi-1)/\phi]W(h,f)$

   $[d(W(h,f)N(h,f))] = [(\phi-1)/\phi]W(h,f)dN(h,f) < W(h,f)dN(h,f)$ (< would be = if W(h,f) didn't fall)

   So, the RHS of (9)′ is just the utility of spending the proceeds.

2. The distortion created by monopolistic competition:

   When monopolistic wage setters increase work, the wage bill goes up less than 1 for 1 since the W(h,f) falls. $\mu = \phi/(\phi-1) > 1$ is the distortion (or markup) created by monopolistic competition. It makes households work too little. As $\phi \to \infty$ (and $\mu \to 1$), the demand curves become infinitely elastic, and the distortion is eliminated, leaving the private marginal benefit of work equal to the marginal cost of work.

3. As G&K (2001) have noted with respect to price markups, there is an analogy between the $1/\mu$ and the income tax; either leads to too little work effort. More later.

If (h,f) works at a fixed-wage firm:

$\mu_w E_{t-1}[g'(\cdot)(N_{s,t}(h,f)/W_{s,t}(h,f)] = E_{t-1}[(u'(\cdot)/P_t)N_{s,t}(h,f)]$ or

$(9)_{fixed}$ $W_{s,t}(h,f) = \mu_w E_{t-1}[g'(\cdot)N_{s,t}(h,f)]/E_{t-1}[(u'(\cdot)/P_t)N_{s,t}(h,f)]$

$= \mu_w E_{t-1}[A_t N_{s,t}(h,f)^{1+\chi}/]/E_{t-1}[(1/P_t C_t(h,f))N_{s,t}(h,f)]$

Firms:

The present value of firm-f's nominal profits stream $\{R_\tau(f)\}$ is:

$PV_t(f) = E_t\sum_{\tau=t}^{\infty}\delta_{t,\tau}R_\tau(f),$

Firms maximize their market value, PV; multiplying by λ_t and using (6), we can write their objective function as:

(10) $MV_t \equiv \lambda_t PV_t(f) = E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t}\lambda_\tau R_\tau(f)$

Recall:

Firm-f's Production Function: $Y_{s,\tau}(f) = Z_{s,\tau}N_{s,\tau}(f)$

Demand (of the bundler) for Firm-f's product: $Y^d_{s,\tau}(f) = (P_{s,\tau}(f)/P_{s,\tau})^{-\theta}Y_{s,\tau}$

Firm-f chooses $P_t(f)$ to max its MV_t:

$$MV_t = E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t}\lambda_\tau R_\tau(f) = E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t}\lambda_\tau[P_{s,\tau}(f)Y_{s,\tau}(f) - W_{s,\tau}(f)Y_{s,\tau}(f)/Z_{s,\tau}]$$

$$= E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t}\lambda_\tau[P_{s,\tau}(f)(P_{s,\tau}(f)/P_{s,\tau})^{-\theta}Y_{s,\tau} - (W_{s,\tau}(f)/Z_{s,\tau})(P_{s,\tau}(f)/P_{s,\tau})^{-\theta}Y_{s,\tau}]$$

$$= E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t}\lambda_\tau Y_{s,\tau}[P_{s,\tau}(f)^{1-\theta}(1/P_{s,\tau})^{-\theta} - (W(f)_{s,\tau}/Z_{s,\tau})(P_{s,\tau}(f)^{-\theta}(1/P_{s,\tau})^{-\theta}]$$

if f is a flexible price firm (letting $\mu_p \equiv \theta/(\theta-1) > 1$):

$P_{s,t}(f)$: $\lambda_t Y_{s,t}[(1-\theta)P_{s,t}(f)^{-\theta}(1/P_{s,t})^{-\theta} + \theta(W_{s,t}(f)/Z_{s,t})P_{s,t}(f)^{-1-\theta}(1/P_{s,t})^{-\theta}] = 0$

$\lambda_t Y_{s,t}(f) = \mu_p\lambda_t Y_{s,t}(f)(W_{s,t}(f)/Z_{s,t})P_{s,t}(f)^{-1}$

canceling the $\lambda_t Y_{s,t}(f)$ (which we can't do in the fixed price case)

$(11)_{flex}$ $P_{s,t}(f) = \mu_p[W_{s,t}(f)/Z_{s,t}]$

If f is a fixed price firm (letting $\mu_p \equiv \theta/(\theta\text{-}1) > 1$):

$P_{s,t}(f)$: $E_{t-1}[\lambda_t Y_{s,t}(f)] = \mu_p E_{t-1}[\lambda_t Y_{s,t}(f)(W_{s,t}(f)/Z_{s,t})P_{s,t}(f)^{-1}]$

$E_{t-1}[(u'(\cdot)/P_t)Y_{s,t}(f)] = \mu_p E_{t-1}[(u'(\cdot)/P_t)Y_{s,t}(f)(W_{s,t}(f)/Z_{s,t})P_{s,t}(f)^{-1}]$

$(11)_{fixed}$ $P_{s,t}(f) = \mu_p E_{t-1}[(u'(\cdot)/P_t)Y_{s,t}(f)(W_{s,t}(f)/Z_{s,t})]/E_{t-1}[(u'(\cdot)/P_t)Y_{s,t}(f)]$

$= \mu_p E_{t-1}[(1/P_t C_t(h,f))Y_{s,t}(f)(W_{s,t}(f)/Z_{s,t})]/E_{t-1}[(1/P_t C_t(h,f))Y_{s,t}(f)]$ (with log utility)

Remark:

If the firm is setting $P_{s,t}(f)$ for multiple periods – as in Taylor or Calvo contracts – we get an obvious generalization of $(16)_{fixed}$; see CCD (2002b), The "New-Keynesian" Phillips Curve.

A Fundamental Relationship between Work and Liesure:

Note: $(9)_{fixed}$, $(11)_{fixed}$ and (2c) $\Rightarrow$

$P_{s,t} = \mu_p E_{t-1}[(u'(\cdot)/P_t)Y_{s,t}(W_{s,t}/Z_{s,t})]/E_{t-1}[(u'(\cdot)/P_t)Y_{s,t}]$

$\Rightarrow$ $E_{t-1}[(u'(\cdot)/P_t)P_{s,t}Y_{s,t}] = \mu_p E_{t-1}[(u'(\cdot)/P_t)W_{s,t}N_{s,t})]$

$W_{s,t} = \mu_w E_{t-1}[g'(\cdot)N_{s,t}]/E_{t-1}[(u'(\cdot)/P_t)N_{s,t}]$

$\Rightarrow$ $E_{t-1}[(u'(\cdot)/P_t)W_{s,t}N_{s,t}] = \mu_w E_{t-1}[g'(\cdot)N_{s,t}]$

So, $E_{t-1}[(u'(\cdot)/P_t)P_{s,t}Y_{s,t}] = \mu E_{t-1}[g'(\cdot)N_{s,t}]$

Using constant expenditure shares: $\mu E_{t-1}[g'(\cdot)N_{s,t}] = E_{t-1}[(u'(\cdot)/P_t)P_{s,t}Y_{s,t}] = E_{t-1}[(u'(\cdot)/P_t)P_t C_t/S]$

Or finally: $\mu E_{t-1}[g'(\cdot)N_{s,t}] = E_{t-1}[(u'(\cdot)C_{s,t}(h,f)]$

Thus we have:

Lemma 1: Relationship between the (expected) utility of consumption and disutility of work –

(A) $(16)_{fixed}$, $(17)_{fixed}$ and (2c) $\Rightarrow$ $E_{t-1}[(u'(\cdot)C_t(h,f)] = \mu E_{t-1}[g'(\cdot)N_{s,t}]$

(B) $u(C) = (1-\gamma)^{-1}C^{1-\gamma}$ and $g(N) = A_t(1+\chi)^{-1}N^{(1+\chi)}$ $\Rightarrow$ $E_{t-1}[u(\cdot)] = [(1+\chi)/(1-\gamma)]\mu E_{t-1}[g(\cdot)]$

(C) $u(C) = \log(C)$ and $g(N) = A_t(1+\chi)^{-1}N^{(1+\chi)}$ $\Rightarrow$ $E_{t-1}[g(\cdot)] = [1/\mu(1+\chi)]$

Since the first order conditions for households and firms in a given sector are identical, we look for an equilibrium in which:

$$N_{s,\tau}(f) = [\int_0^1 N_{s,\tau}(h,f)^{(\phi-1)/\phi}dh]^{\phi/(\phi-1)} = [N_{s,\tau}(h,f)^{(\phi-1)/\phi}(\int_0^1 dh)]^{\phi/(\phi-1)} = N_{s,\tau}(h,f)$$

$$W_{s,\tau}(f) = [\int_0^1 W_{s,\tau}(h,f)^{1-\phi}dh]^{1/(1-\phi)} = [W_{s,\tau}(h,f)^{1-\phi}(\int_0^1 dh)]^{1/(1-\phi)} = W_{s,\tau}(h,f)$$

So, we can drop the indices (f) and the (h,f), and let:

$$N_{s,t}(h,f) = N_{s,t}(f) \equiv N_{s,t} \text{ and } W_{s,t}(h,f) = W_{s,t}(f) \equiv W_{s,t}$$

Similarly:

$$Y_{s,t}(f) = Y_{s,t} \text{ and } P_{s,t}(f) = P_{s,t}$$

Aggregation: (Recall, there are S sectors, and measure S households)

$$\int_1^{S+1}[\int_0^1 C_{s,t}(h,f)dh]df = SC_{s,t}(h,f) \equiv SC_{s,t} = Y_{s,t} \equiv Y_{s,t}(f) = [\int_s^{s+1} Y_{s,t}(f)^{(\theta-1)/\theta}df]^{\theta/(\theta-1)}$$

$$\int_1^{S+1}[\int_0^1 C_t(h,f)dh]df = SC_t(h,f) \equiv SC_t = Y_t \equiv \Pi_{s=1}^{S} Y_{s,t}{}^{1/S}$$

An Equilibrium is characterized by –

In equilibrium, the wage and price equations become (for s = 1, 2, ... , S):

$(12)_{flex}$ $W_{s,t}/P_t = \mu_w[g'(N_{s,t})/u'(C_t)]$

$(12)_{fixed}$ $W_{s,t} = \mu_w E_{t-1}[g'(N_{s,t})N_{s,t}]/E_{t-1}[(u'(C_t)/P_t)N_{s,t}]$

$(13)_{flex}$ $P_{s,t} = \mu_p[W_{s,t}/Z_{s,t}]$

$(13)_{fixed}$ $P_{s,t} = \mu_p E_{t-1}[(u'(C_t)/P_t)Y_{s,t}(W_{s,t}/Z_{s,t})]/E_{t-1}[(u'(C_t)/P_t)Y_{s,t}]$

The other equilibrium conditions become:

(14) $P_tC_t = SP_{s,t}C_{s,t}$

(15) $P_t = S\Pi_{s=1}^{S} P_{s,t}{}^{1/S}$

(16) $I_t^{-1} = E_t[\delta_{t,t+1}] = \beta E_t[(u'(C_{t+1})/u'(C_t))(P_t/P_{t+1})]$

(17) $V_t v'(M_t/P_t) = u'(C_t)[1 - I_t^{-1}]$

Monetary policy procedures:

(18) $P_tC_t \equiv \Omega(A_t, Z_{1,t}, \ldots, Z_{S,t})$

The flexible wage/price "benchmark" –

$(12)_{flex}$ and (14) $\Rightarrow$ $W_{s,t} = \mu_w A_t N_{s,t}^{\chi} P_t C_t = \mu_w S A_t N_{s,t}^{\chi} P_{s,t} C_{s,t}$

$\Rightarrow$ $W_{s,t}/P_{s,t} = \mu_w S A_t N_{s,t}^{\chi} C_{s,t} = \mu_w A_t N_{s,t}^{\chi} Y_{s,t} = \mu_w A_t N_{s,t}^{\chi} N_{s,t} Z_{s,t} = \mu_w A_t Z_{s,t} N_{s,t}^{1+\chi}$

$(13)_{flex}$ $\Rightarrow$ $W_{s,t}/P_{s,t} = (1/\mu_p) Z_{s,t}$ $\Rightarrow$ $N_{s,t} = \mu^{-1/(1+\chi)} A_t^{-1/(1+\chi)}$ where $\mu = \mu_p \mu_w$ for all s $\Rightarrow$ work equalizes

(18) $\Rightarrow W_{s,t} = \mu_w A_t N_{s,t}^{\chi} P_t C_t = \mu_w A_t N_{s,t}^{\chi} \Omega_t = \mu_w \mu^{-\chi/(1+\chi)} A_t^{1/(1+\chi)} \Omega_t$ for all s $\Rightarrow$ wages equalize

And for any two sectors s and s′, $(13)_{flex}$ $\Rightarrow$

$P_{s,t}/P_{s',t} = \mu_p[W_{s,t}/Z_{s,t}]/\mu_p[W_{s',t}/Z_{s',t}] = Z_{s',t}/Z_{s,t}$

$P_{s',t}/P_t = S Z_{s',t}^{-1} \Pi_{s=1}^{S} Z_{s,t}^{1/S}$

and the aggregate price level becomes:

$P_t C_t = \Omega_t$ $\Rightarrow$ $P_t = \Omega_t / C_t = \Omega_t / \Pi_{s=1}^{S} C_{s,t}^{1/S}$

Lemma 2: *The Flexible Wage/Price Solution –*

If $u(C_t) = \log(C_t)$ and $g(N_t) = A_t(1+\chi)^{-1} N_t^{1+\chi}$, then

(A) $P_{s,t}^* = \mu_p (W_{s,t}^*/Z_{s,t})$

(B) $N_{s,t}^* = \mu^{-1/(1+\chi)} A_t^{-1/(1+\chi)}$ and

(C) $Y_{s,t}^* = Z_{s,t} N_{s,t}^* = \mu^{-1/(1+\chi)} A_t^{-1/(1+\chi)} Z_{s,t} = S C_{s,t}^*$

(D) $C_{s,t}^*/C_{s',t}^* = P_{s',t}^*/P_{s,t}^* = Z_{s,t}/Z_{s',t}$ and $P_{s',t}^*/P_t^* = (\Pi_{s=1}^{S} Z_{s,t}^{1/S})/S Z_{s',t}$

(E) $C_t^* = (1/S)\mu^{-1/(1+\chi)} A_t^{-1/(1+\chi)} \Pi_{s=1}^{S} Z_{s,t}^{1/S}$

(F) $P_t^* = \Omega_t S \mu^{1/(1+\chi)} A_t^{1/(1+\chi)} \Pi_{s=1}^{S} Z_{s,t}^{-1/S}$ and $W_{s,t}^* = \mu_w \mu^{-\chi/(1+\chi)} A_t^{1/(1+\chi)} \Omega_t$

where *'s denote flexible wage/price values and $\mu \equiv \mu_p \mu_w$ is the combined markup.

Derivation of labor supply and demand for sector s:

Labor demand: $W_{s,t}^*/P_{s,t}^* = (1/\mu_p) Z_{s,t}$

$W_{s,t}/P_t = \mu_w[g'(\cdot)/u'(\cdot)] = \mu_w[A_t N_{s,t}^{\chi}]/C_t^{-\gamma} = \mu_w[A_t N_{s,t}^{\chi}](C_{s,t}(SP_{s,t}/P_t))^{\gamma}$

$(W_{s,t}/P_t)(P_t/P_{s,t})^{\gamma} = \mu_w[A_t N_{s,t}^{\chi}](Y_{s,t})^{\gamma} S^{\gamma-1} = \mu_w[A_t N_{s,t}^{\chi}](N_{s,t} Z_{s,t})^{\gamma} S^{\gamma-1} = \mu_w A_t N_{s,t}^{\gamma+\chi} Z_{s,t}^{\gamma} S^{\gamma-1}$

$(W_{s,t}/P_t)(P_t/P_{s,t})^{\gamma} = (W_{s,t}/P_{s,t})(P_t/P_{s,t})^{\gamma-1} = \mu_w A_t N_{s,t}^{\gamma+\chi} Z_{s,t}^{\gamma} S^{\gamma-1}$

Labor Supply: $W_{s,t}^*/P_{s,t}^* = \mu_w A_t (N_{s,t}^*)^{\gamma+\chi} Z_{s,t}^{\gamma} (P_{s,t}/P_t)^{\gamma-1} S^{\gamma-1}$

Taking logs of each curve (and letting small letters denote logs), we have:

$w_{s,t}^* - p_{s,t}^* = \log(1/\mu_p) + z_{s,t}$

$w_{s,t}^* - p_{s,t}^* = \text{constant} + (\gamma+\chi) n_{s,t}^* + a_t + \{\gamma + (\gamma-1)[(1/S) - 1]\} z_{s,t} + (1-\gamma)(1/S)\sum_{s' \neq s} z_{s,t}$

$= \text{constant} + (\gamma+\chi) n_{s,t}^* + a_t + [1 + (\gamma-1)(1/S)] z_{s,t} + (1-\gamma)(1/S)\sum_{s' \neq s} z_{s,t}$

Solution for sectors with sticky prices (and sticky or flexible wages):

Fixed expenditure shares $\Rightarrow$ $\Omega_t = P_{s,t}Y_{s,t}$

Lemma 1 $\Rightarrow$ $E_{t-1}[g(\cdot)] = [1/\mu(1+\chi)]$ $\Rightarrow$ $E_{t-1}[A_tN_{s,t}^{\psi}] = (1/\mu)$ where $\psi \equiv 1+\chi$

$\Rightarrow$ $E_{t-1}[A_t(Y_{s,t}/Z_{s,t})^{\psi}] = (1/\mu)$ $\Rightarrow$ $E_{t-1}[A_t(\Omega_t/P_{s,t}Z_{s,t})^{\psi}] = 1/\mu$

$\Rightarrow$ $P_{s,t} = \mu^{1/\psi}\{E_{t-1}[A_t(\Omega_t/Z_{s,t})^{\psi}]\}^{1/\psi}$

Recall: $Y_{s,t}^* = \mu^{-1/\psi}A_t^{-1/\psi}Z_{s,t}$

$P_{s,t}^* = \Omega_t/Y_{s,t}^* = \Omega_t\mu^{1/\psi}A_t^{1/\psi}/Z_{s,t}$

Assuming log-normality:

$$p_{s,t} = (1/\psi)\log(\mu) + (1/\psi)\log E_{t-1}[A_t(\Omega_t/Z_{s,t})^{\psi}]$$

$$= (1/\psi)\log(\mu) + (1/\psi)E_{t-1}[a_t + \psi(\omega_t - z_{s,t})] + ½(1/\psi)VAR_{t-1}[a_t + \psi(\omega_t - z_{s,t})]$$

$$= (1/\psi)\log(\mu) + E_{t-1}[\omega_t - z_{s,t} + a_t/\psi] + ½\psi VAR_{t-1}[\omega_t - z_{s,t} + a_t/\psi]$$

$$= p_{s,t}^* - \{\omega_t - E_{t-1}[\omega_t]\} + \{z_{s,t} - E_{t-1}[z_{s,t}]\} - \{(a_t/\psi) - E_{t-1}[a_t/\psi]\} + ½\psi VAR_{t-1}[\omega_t - z_{s,t} - a_t/\psi]$$

$$y_{s,t} - y_{s,t}^* = -(p_{s,t} - p_{s,t}^*)$$

$$= \{\omega_t - E_{t-1}[\omega_t]\} - \{z_{s,t} - E_{t-1}[z_{s,t}]\} + \{(a_t/\psi) - E_{t-1}[a_t/\psi]\} - ½\psi VAR_{t-1}[\omega_t - z_{s,t} + a_t/\psi]$$

Solution for sectors with sticky wages and flexible prices:

Fixed expenditure shares $\Rightarrow$ $\Omega_t = P_{s,t}*Y_{s,t} = \mu_p[W_{s,t}/Z_{s,t}]Y_{s,t} = \mu_p W_{s,t}N_{s,t}$

Lemma 1 $\Rightarrow$ $E_{t-1}[g(\cdot)] = [1/\mu(1+\chi)]$ $\Rightarrow$ $E_{t-1}[A_tN_{s,t}^{\psi}] = 1/\mu$ where $\psi \equiv 1+\chi$

$\Rightarrow E_{t-1}[A_tN_{s,t}^{\psi}] = (1/\mu_p)^{\psi}E_{t-1}[A_t(\Omega_t/W_{s,t})^{\psi}] = 1/\mu$

$\Rightarrow W_{s,t} = (\mu^{1/\psi}/\mu_p)\{E_{t-1}[A_t\Omega_t^{\psi}]\}^{1/\psi}$

Recall: $Y_{s,t}* = \mu^{-1/\psi}A_t^{-1/\psi}Z_{s,t}$ or $N_{s,t}* = (1/\mu)^{1/\psi}A_t^{-1/\psi}$

$W_{s,t}* = (1/\mu_p)(\Omega_t/N_{s,\tau}*) = (\mu^{1/\psi}/\mu_p)A_t^{1/\psi}\Omega_t$

Assuming log-normality:

$w_{s,t} = \log(\mu^{1/\psi}/\mu_p) + (1/\psi)\log E_{t-1}[A_t\Omega_t^{\psi}]$

$= \log(\mu^{1/\psi}/\mu_p) + (1/\psi)E_{t-1}[\psi\omega_t + a_t] + ½(1/\psi)VAR_{t-1}[\psi\omega_t + a_t]$

$= \log(\mu^{1/\psi}/\mu_p) + E_{t-1}[\omega_t + a_t/\psi] + ½\psi VAR_{t-1}[\omega_t + a_t/\psi]$

$= w_{s,t}* - \{\omega_t - E_{t-1}[\omega_t]\} - \{(a_t/\psi) - E_{t-1}[a_t/\psi]\} + ½\psi VAR_{t-1}[\omega_t + a_t/\psi]$

$n_{s,t} - n_{s,t}* = -(w_{s,t} - w_{s,t}*) = \{\omega_t - E_{t-1}[\omega_t]\} + \{(a_t/\psi) - E_{t-1}[a_t/\psi]\} - ½\psi VAR_{t-1}[\omega_t + a_t/\psi]$

Lemma 3: Sticky wage/price solutions –

In sectors where wages are sticky and prices are flexible:

A. $y_{s,t} - y_{s,t}* = \{\omega_t - E_{t-1}[\omega_t]\} + \{(a_t/\psi) - E_{t-1}[a_t/\psi]\} - ½\psi VAR_{t-1}[\omega_t + a_t/\psi] = n_{s,t} - n_{s,t}*$

B. $w_{s,t} = w_{s,t}* - \{\omega_t - E_{t-1}[\omega_t]\} - \{(a_t/\psi) - E_{t-1}[a_t/\psi]\} + ½\psi VAR_{t-1}[\omega_t + a_t/\psi]$

In sectors where prices are sticky and wages are either flexible or sticky:

C. $y_{s,t} - y_{s,t}* = \{\omega_t - E_{t-1}[\omega_t]\} - \{z_{s,t} - E_{t-1}[z_{s,t}]\} + \{(a_t/\psi) - E_{t-1}[a_t/\psi]\} - ½\psi VAR_{t-1}[\omega_t - z_{s,t} + a_t/\psi]\}$

D. $p_{s,t} = p_{s,t}* - \{\omega_t - E_{t-1}[\omega_t]\} + \{z_{s,t} - E_{t-1}[z_{s,t}]\} - \{(a_t/\psi) - E_{t-1}[a_t/\psi]\} + ½\psi VAR_{t-1}[\omega_t - z_{s,t} + a_t/\psi]\}$

where $y_{s,t}*$ are flex-price levels of output, and $w_{s,t}*$ and $p_{s,t}*$ are "notional" prices (or "shadow" prices) of labor and output; $\psi \equiv 1+\chi$.

We assume the goal of monetary policy is to maximize average household utility:

(19) $W_t \equiv E_{t-1}\sum_{\tau=t}^{\infty}\beta^{\tau-t}[u(C_t) - (1/S)\sum_{s=1}^{S}g(N_{s,\tau})]$

Proposition 1: Let $u(C) \equiv \log(C)$ and $g(N) \equiv \psi^{-1}AN^{\psi}$; let $Z_{s,t}$ have a log-normal distribution; let W be the set of sectors that have fixed wages and flexible prices, and let P be the set of sectors that have sticky prices and flexible or sticky wages; and let $g_{s,t} \equiv y_{s,t} - y_{s,t}^*$ be the "output gap" in sector s. Then, the goal of monetary policy is to choose a rule, $\omega(\cdot)$, that maximizes $\sum_{s=1}^{S}E_{t-1}c_{s,t}$; moreover, $\sum_{s=1}^{S}E_{t-1}c_{s,t}$ can be expressed in three ways:

(A) $\sum_{s=1}^{S}E_{t-1}c_{s,t} = -S\log(S) + \sum_{s=1}^{S}E_{t-1}y_{s,t}^* - ½\sum_{s\in W}VAR_{t-1}[\omega_t + a_t/\psi]$
$- ½\sum_{s\in P}VAR_{t-1}[\omega_t - z_{s,t} + a_t/\psi]$

(B) $\sum_{s=1}^{S}E_{t-1}c_{s,t} = -S\log(S) + \sum_{s=1}^{S}E_{t-1}y_{s,t}^* - ½\sum_{s\in W}VAR_{t-1}[g_{s,t}] - ½\sum_{s\in P}VAR_{t-1}[g_{s,t}]$

(C) $\sum_{s=1}^{S}E_{t-1}c_{s,t} = -S\log(S) + \sum_{s=1}^{S}E_{t-1}y_{s,t}^* - ½\sum_{s\in W}VAR_{t-1}[w_{s,t}^*] - ½\sum_{s\in P}VAR_{t-1}[p_{s,t}^*]$

Proof:

Lemma 1 $\Rightarrow$ $W_t = E_{t-1}\sum_{\tau=t}^{\infty}\beta^{\tau-t}[\log(C_t) - (\mu\psi)^{-1}]$ and the solution and the maximization are static; since monetary policy can not affect the expected disutility of work, the goal reduces to maximizing $E_{t-1}\log(C_t)$, which by the fixed expenditure shares is proportional to $\sum_{s=1}^{S}E_{t-1}c_{s,t}$. The expressions for $\sum_{s=1}^{S}E_{t-1}c_{s,t}$ follow directly from Lemma 3.

For convenience, we re-write Lemma 2 in log terms:

Lemma 2a: The benchmark *flexible wage/price solution* is:

(A) $p_{s,t}^* = \log(\mu_p) + w_{s,t}^* - z_{s,t}$

(B) $n_{s,t}^* = -\psi^{-1}\log(\mu) - \psi^{-1}a_t$ and $w_{s,t}^* = \log(\mu_w) - \chi\psi^{-1}\log(\mu) + \psi^{-1}a_t + \omega_t$ (for all s)

(C) $y_{s,t}^* = z_{s,t} + n_{s,t}^* = -\psi^{-1}\log(\mu) - \psi^{-1}a_t + z_{s,t} = c_{s,t}^* + \log(S)$

(D) $p_{s',t}^* - p_{s,t}^* = z_{s,t} - z_{s',t}$ and $p_{s',t}^* - p_t^* = -\log(S) - z_{s,t} + (1/S)\sum_{s=1}^{S}z_{s',t}$

(E) $c_{s,t}^* - c_{s',t}^* = z_{s,t} - z_{s',t}$

(F) $c_t^* = -\psi^{-1}\log(\mu) - \psi^{-1}a_t + (1/S)\sum_{s=1}^{S}z_{s,t} - \log(S)$ and
$p_t^* = \log(S) + \psi^{-1}\log(\mu) + \psi^{-1}a_t - (1/S)\sum_{s=1}^{S}z_{s,t}$
where $\mu = \mu_p\mu_w$ is the combined markup

Inefficiency:

Efficiency requires that the marginal product of labor (MPL) in any sector, s, be equal to the marginal rate of substitution (MRS) between leisure and the good produced in sector s. Dropping time subscripts, the efficiency condition is:

$$MRS = \frac{g'(N_s)}{u'(C)\left(\partial C / \partial C_s\right)} = \left[\frac{g'(.)}{u'(.)}\right]\left(\frac{SC_s}{C}\right) = Z_s = MPL$$

So, the efficiency condition can be expressed as

$$g'(\cdot)/u'(\cdot) = Z_s(C/SC_s)$$

To see how the monopoly distortions in the labor and goods markets are sources of inefficiency in our model, note that the wage setting equation $(9)_{flex}$ and constant expenditure shares (14) imply

$$g'(\cdot)/u'(\cdot) = (1/\mu_w)(W_s/P) = (1/\mu_w)(W_s/P_s)(P_s/P) = (1/\mu_w)(W_s/P_s)(C/SC_s)$$

Then use the price setting equation $(11)_{flex}$ to get

$$g'(\cdot)/u'(\cdot) = (1/\mu_w \mu_p)Z_s(C/SC_s) = (1/\mu)Z_s(C/SC_s) < Z_s(C/SC_s)$$

Proposition 7 (Benigno): Suppose a currency union consists of the two countries, A and B, described above. Then,

A. there is no full information policy rule that can achieve the optimal flexible wage/price solution union wide;

B. the optimal policy rule sets $VAR_{t-1}[\omega_t - (1/3)z_{a,t} - (2/3)z_{b,t} + a_t/\psi] = 0$

C. the optimal policy can also be characterized as

$VAR_{t-1}[(1/3)p_{ap,t}* + (2/3)p_{bp,t}*] = VAR_{t-1}[(1/3)p_{af,t} + (2/3)p_{bf,t}] = 0.$

Proof:

A. This result, like Proposition 6, follows directly from Lemma 3.

B. From Proposition 1, Part A, the optimal monetary policy minimizes

$$J = VAR_{t-1}[\ \omega_t - z_{a,t} + a_t/\psi] + 2VAR_{t-1}[\omega_t - z_{b,t} + a_t/\psi]$$

Expanding the variance terms, we have:

$$\begin{aligned} J &= 3VAR_{t-1}[\ \omega_t + a_t/\psi] + VAR_{t-1}[\ z_{a,t}\ ] + 2VAR_{t-1}[\ z_{b,t}\ ] \\ &\quad - 2COV_{t-1}[\ \omega_t + a_t/\psi\ ,\ z_{a,t}\ ] - 4COV_{t-1}[\ \omega_t + a_t/\psi\ ,\ z_{b,t}\ ] \\ &= 3VAR_{t-1}[\ \omega_t + a_t/\psi] + VAR_{t-1}[\ z_{a,t}\ ] + 2VAR_{t-1}[\ z_{b,t}\ ] \\ &\quad - 6COV_{t-1}[\ \omega_t + a_t/\psi\ ,\ z_{a,t}/3\ ] - 6COV_{t-1}[\ \omega_t + a_t/\psi\ ,\ 2z_{b,t}/3\ ] \\ &= 3VAR_{t-1}[\ \omega_t + a_t/\psi] + VAR_{t-1}[\ z_{a,t}\ ] + 2VAR_{t-1}[\ z_{b,t}\ ] \\ &\quad - 6COV_{t-1}[\ \omega_t + a_t/\psi\ ,\ z_{a,t}/3 + 2z_{b,t}/3\ ] \\ &= 3VAR_{t-1}[\ \omega_t + a_t/\psi - z_{a,t}/3 - 2z_{b,t}/3\ ] + VAR_{t-1}[\ z_{a,t}\ ] + 2VAR_{t-1}[\ z_{b,t}\ ] \\ &\quad - 3VAR_{t-1}[\ z_{a,t}/3 + 2z_{b,t}/3\ ] \\ &= 3VAR_{t-1}[\ \omega_t + a_t/\psi - z_{a,t}/3 - 2z_{b,t}/3\ ] + (2/3)VAR_{t-1}[\ z_{a,t}\ ] + (2/3)VAR_{t-1}[\ z_{b,t}\ ] \\ &\quad - (4/3)COV_{t-1}[\ z_{a,t}/3\ ,\ 2z_{b,t}/3\ ] \\ &= 3VAR_{t-1}[\ \omega_t + a_t/\psi - z_{a,t}/3 - 2z_{b,t}/3\ ] + (2/3)VAR_{t-1}[\ z_{a,t} - z_{b,t}\ ] \end{aligned}$$

Since the second variance term in the last expression does not depend on ω_t, the optimal policy minimizes J by setting the first variance term equal to zero.

C. We have

ω_t - (1/3)$z_{a,t}$ - (2/3)$z_{b,t}$ + a_t/ψ = (1/3)[ω_t - $z_{a,t}$ + a_t/ψ] + (2/3)[ω_t - $z_{b,t}$ + a_t/ψ]

= (1/3)[ω_t - $y_{ap,t}*$ - $\psi^{-1}\log(\mu)$] + (2/3)[ω_t - $y_{bp,t}*$ - $\psi^{-1}\log(\mu)$]

= (1/3)$p_{ap,t}*$ + (2/3)$p_{bp,t}*$ - $\psi^{-1}\log(\mu)$

(where we have used $\omega_t = p_{s,t}* + y_{s,t}*$)

So, B implies C.

Note that C says that the optimal policy eliminates uncertainty about a union-wide price index in which the country weights (one-third and two-thirds) depend on the sizes of sticky price sectors.

Proposition 9 (Erceg, Henderson and Levin): Suppose the economy consists of a fixed wage sector (denoted by “w”), a sticky price sector (denoted by “p”), and a flexible wage/price sector (denoted by “f”). Suppose productivity shocks are perfectly correlated across sectors ($z_{p,t} = z_{w,t} = z_{f,t} \equiv z_t$). Then,

A. There is no rule, $\omega(a_t, z_t)$, that makes $y_{p,t} = y_{p,t}*$ and $y_{w,t} = y_{w,t}*$ for all values of z_t.

B. A policy that targets the aggregate price level (makes $p_t = p_t^T$) implies $y_{p,t} = y_{p,t}*$.

C. A policy that targets the aggregate wage rate (makes $w_t = w_t^T$) implies $y_{w,t} = y_{w,t}*$.

D. The optimal policy rule sets $VAR_{t-1}[\omega_t - (1/2)z_t + a_t/\psi] = 0$

E. The optimal policy can also be characterized as

$VAR_{t-1}[(1/2)p_{p,t}* + (1/2)w_{w,t}*] = VAR_{t-1}[(1/2)p_{f,t} + (1/2)w_{f,t}] = 0$

Proof:

A. Lemma 3 implies that: (1) to get the fixed price sector right, ω_t, must respond to both a_t and z_t, and (2) to get the fixed wage sector right, ω_t, must respond to a_t alone.

B. The proof is similar to the proof of Proposition 3.

C. The proof is similar to the proof of Proposition 7.

D. From Proposition 1, Part A, the optimal monetary policy minimizes

$J = VAR_{t-1}[\ \omega_t - z_t + a_t/\psi] + VAR_{t-1}[\omega_t + a_t/\psi]$. Expanding the first variance term, we have $J = 2VAR_{t-1}[\ \omega_t + a_t/\psi] + VAR_{t-1}[\ z_t\ ] - 2COV_{t-1}[\ \omega_t + a_t/\psi\ ,\ z_t\ ]$

$$= 2VAR_{t-1}[\ \omega_t + a_t/\psi] + VAR_{t-1}[\ z_t\ ] - 4COV_{t-1}[\ \omega_t + a_t/\psi\ ,\ z_t/2\ ]$$

$$= 2VAR_{t-1}[\ \omega_t + a_t/\psi - z_t/2\ ] + (1/2)VAR_{t-1}[\ z_t\ ]$$

Since the second variance term does not depend on ω_t, optimal policy minimizes J by setting the first variance term equal to zero.

E. We have

$$\omega_t - (1/2)z_t + a_t/\psi = (1/2)[\omega_t - z_t + a_t/\psi] + (1/2)[\omega_t + a_t/\psi]$$

$$= (1/2)[\omega_t - y_{p,t}^* - \psi^{-1}\log(\mu)] + (1/2)[\omega_t - y_{w,t}^* + z_t - \psi^{-1}\log(\mu)]$$

$$= (1/2)p_{p,t}^* + (1/2)[w_{w,t}^* + \log\mu_p] - \psi^{-1}\log(\mu)$$

(where we have used $\omega_t = p_{s,t}^* + y_{s,t}^*$ and $p_{s,t}^* = \log(\mu_p) + w_{s,t}^* - z_{s,t}$)

So, $VAR_{t-1}[\omega_t - (1/2)z_t + a_t/\psi] = 0$

$$\Rightarrow\ VAR_{t-1}[(1/2)p_{p,t}^* + (1/2)w_{w,t}^*] = 0$$