

Appendix I
Interest Rate Rules and Price Determinacy:
the Role of Transactions Services of Bonds
by
Matthew Canzoneri and Behzad Diba

This appendix completes the proof of Proposition 1. We show that the non-steady-state solutions of equations (5a) to (5c) and (9) are not bounded, and we discuss whether or not these explosive paths violate the household's transversality condition. Consider first a potential equilibrium starting at date t with a positive price level below the steady-state value: $0 < P_t < \bar{P}$. Given this price level, there is a unique value of the money supply that satisfies (5a). To see this, write (5a) as:

$$(I.1) \quad \frac{M_t}{P_t} + k\left(\frac{\bar{L}}{P_t} - \frac{M_t}{P_t}\right) = 1.$$

Note that the left-hand side of (I.1) starts below 1 when $M_t = 0$ (because $k(\cdot) < 1$), is continuous and strictly increasing in M_t (because $k'(\cdot) < 1$), and goes above 1 when M_t reaches its steady-state value \bar{M} (because \bar{M} satisfies (5a) at the higher price level \bar{P}). So for (I.1) to hold, we must have $0 < M_t < \bar{M}$. Since $M_t + B_t = \bar{M} + \bar{B}$, we must also have $B_t > \bar{B}$. And since $k'(\cdot)$ is strictly monotone decreasing, we get $k'(\frac{B_t}{P_t}) < k'(\bar{b})$.

Now, consider (5c). Since i_t is fixed at \bar{i} both in and out of the steady state, we must have

$$\bar{i} < i_t^* < \bar{i}^* = \beta^{-1} - 1. \quad \text{The first inequality holds because } k'(\cdot) \text{ is positive, and the second one}$$

holds because $k'(\cdot)$ is smaller out of the steady state. It then follows from (5b) that $\beta(1 + \bar{i}) < \Pi_t < \bar{\Pi} = 1$. That is, a price path starting below the steady-state value must be deflationary, but Π_t

is bounded above β as long as \bar{i} is positive (which we assume throughout).

The deflationary path ($P_{t+1} < P_t$) implies that $0 < M_{t+1} < M_t$ and $\beta(1 + \bar{i}) < \Pi_{t+1} < \Pi_t$ by the same reasoning steps that we used above. Iterating on this argument, we see that a potential equilibrium starting with $P_t < \bar{P}$ must involve strictly decreasing (but suitably bounded) money-supply and inflation sequences. From the bounds on the inflation sequence, it follows that for any $j > 0$,

$$(I.2) \quad P_t \beta^j (1 + \bar{i})^j < P_{t+j} < P_t (\Pi_t)^j.$$

As j tends to infinity, (I.2) implies that P_{t+j} tends to zero.

Since consumption is constant in equilibrium, the household's transversality condition is:

$$(I.3) \quad \lim_{T \rightarrow \infty} \beta^T \left(\frac{L_T}{P_T} \right) = 0.$$

The inequality on the left-hand-side of (I.2) implies that the transversality condition is satisfied along the deflationary path characterized above. So, the model of Section III does not rule out divergent deflationary paths.

Consider next a potential equilibrium starting at date t with $P_t > \bar{P}$. Using the same steps as above, such an equilibrium must involve increasing and unbounded price-level and money-supply sequences. We can rule out such inflationary paths if we rule out negative values for bonds. But if we allow the central bank to purchase private bonds (so that the money supply can grow without bound), these inflationary paths also constitute theoretically viable equilibria.

We have relegated this discussion to the appendix because the theoretical possibility of explosive paths in monetary models is a different issue from what we address in the text. If we restrict our analysis to a bounded space (ℓ_∞), the steady-state equilibrium is the only solution to the model of Section III.

Appendix II

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This appendix contains the derivations for the linearized model in Section IV. We use the cash-in-advance constraint (2a), without restricting V to equal one. This allows us to get more plausible steady-state values for the ratios of money and liabilities to GDP, but does not affect the eigenvalues and dynamics of the model. Since real liabilities are the sum of real money and bond holdings, $l_t = m_t + b_t$, (2a) can be expressed as:

$$(II.1) \quad l_t = \frac{1}{V} + b_t - k(b_t)$$

Combining (5b) and (5c), and denoting the gross nominal interest rate as I_t , we have

$$(II.2) \quad I_t = \frac{\Pi_t}{\beta} - \left(\frac{\Pi_t - \beta}{\beta} \right) k'(b_t)$$

The flow budget constraint (14) can be expressed as:

$$(II.3) \quad \Pi_{t-1} l_t = l_{t-1} + \Pi_{t-1} d_t$$

The model is closed with the fiscal and monetary feedback rules in (15) and (16):

$$(II.4) \quad d_t = z_t - \rho l_{t-1}$$

$$(II.5) \quad I_t = A \Pi_{t-1}^0$$

Linearizing equations (II.1) to (II.5) and expressing the variables as percentage deviations

(denoted by a “^” above the variables) from their steady-state values, we get:

$$(II.6) \hat{l}_t = \left(\frac{\bar{b}[1 - k'(\bar{b})]}{\bar{l}} \right) \hat{b}_t$$

$$(II.7) \hat{l}_t = \left(\frac{\bar{\Pi}[1 - k'(\bar{b})]}{\beta \bar{l}} \right) \hat{\Pi}_t - \left(\frac{\bar{b}(\bar{\Pi} - \beta)k''(\bar{b})}{\beta \bar{l}} \right) \hat{b}_t$$

$$(II.8) \hat{l}_t = \left(\frac{1}{\bar{\Pi}} \right) \hat{l}_{t-1} - \left(\frac{\bar{l} - \bar{d}}{\bar{l}} \right) \hat{\Pi}_{t-1} + \left(\frac{\bar{d}}{\bar{l}} \right) \hat{d}_t$$

$$(II.9) \hat{d}_t = \left(\frac{\bar{z}}{\bar{d}} \right) \hat{z}_t - \left(\frac{\rho \bar{l}}{\bar{d}} \right) \hat{l}_{t-1}$$

$$(II.10) \hat{l}_t = \theta \hat{\Pi}_{t-1}$$

Substituting (II.9) in (II.8), and noting that (II.3) implies $\bar{\Pi} = \bar{l} / (\bar{l} - \bar{d})$, we get:

$$(II.11) \hat{l}_t = \left(\frac{1}{\bar{\Pi}} - \rho \right) \hat{l}_{t-1} - \left(\frac{1}{\bar{\Pi}} \right) \hat{\Pi}_{t-1} + \left(\frac{\bar{z}}{\bar{d}} \right) \hat{z}_t$$

(II.2) implies

$$(II.12) \bar{l} = \frac{\bar{\Pi}[1 - k'(\bar{b})]}{\beta} + k'(\bar{b})$$

Substituting (II.6) and (II.12) in (II.7), we get

$$(II.13) \hat{l}_t = \delta_1 \hat{\Pi}_t - \left(\frac{\bar{l}(\bar{\Pi} - \beta)k''(\bar{b})}{[1 - k'(\bar{b})]\{\beta k'(\bar{b}) + \bar{\Pi}[1 - k'(\bar{b})]\}} \right) \hat{l}_t$$

where

$$\delta_1 = \frac{\bar{\Pi}[1 - k'(\bar{b})]}{\bar{\Pi}[1 - k'(\bar{b})] + \beta k'(\bar{b})}$$

Substituting (II.10) and (II.11) in (II.13), we get

$$(II.14) \hat{\Pi}_\tau = \delta_2 \left(\frac{1}{\bar{\Pi}} - \rho \right) \hat{i}_{t-1} + \left(\frac{\theta}{\delta_1} - \frac{\delta_2}{\bar{\Pi}} \right) \hat{\Pi}_{t-1} + \left(\frac{\delta_2 \bar{z}}{1} \right) \hat{z}_t$$

where

$$\delta_2 = \frac{\bar{i}(\bar{\Pi} - \beta)k''(\bar{b})}{[1 - k'(\bar{b})]^2}$$

The system of equations (17) in the text is the homogeneous system for (II.11) and (II.14). The eigenvalues of the system are

$$\mu_1, \mu_2 = \frac{1}{2} \left\{ \delta_3 \pm \sqrt{\delta_3^2 - \left(\frac{4\theta}{\delta_1} \right) \left(\frac{1}{\bar{\Pi}} - \rho \right)} \right\}, \text{ where } \delta_3 = \frac{1 - \delta_2}{\bar{\Pi}} - \rho + \frac{\theta}{\delta_1}$$

The eigenvalues are real for plausible parameter values. To show this, we can rewrite

$$\begin{aligned} & \left(\frac{1 - \delta_2}{\bar{\Pi}} - \rho + \frac{\theta}{\delta_1} \right)^2 - \left(\frac{4\theta}{\delta_1} \right) \left(\frac{1}{\bar{\Pi}} - \rho \right) \\ &= \left(\frac{1}{\bar{\Pi}} - \rho \right)^2 + \left(\frac{\theta}{\delta_1} - \frac{\delta_2}{\bar{\Pi}} \right)^2 - 2 \left(\frac{1}{\bar{\Pi}} - \rho \right) \left(\frac{\theta}{\delta_1} + \frac{\delta_2}{\bar{\Pi}} \right) \\ &= \left(\frac{1 + \delta_2}{\bar{\Pi}} - \rho - \frac{\theta}{\delta_1} \right)^2 - 4\delta_2(1 - \rho\bar{\Pi}) \end{aligned}$$

Since we have $\delta_2 < 0$, the eigenvalues are real as long as $\rho\bar{\Pi} < 1$. This restriction is satisfied for all the cases we consider in Table 1. So, when the steady-state equilibrium is a saddle point, the solution to the linearized system converges to the steady state monotonically.

For our numerical calculations reported in Table 1, we assume $k(b_t) = \kappa[1 - \exp(-b_t)]$.

We set $\kappa = 0.2$ and $V = 5$ throughout and vary the steady-state values of i and z , to get the values for inflation and the budget deficit reported in the Table.

Appendix III
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This appendix completes the proof of Proposition 2. It remains to show that the set of initial interest rates that the central bank cannot choose (in Proposition 2) is a set of measure zero. The only reason this is an issue in our setup is that $k(\cdot)$ is not differentiable at zero; so we need a technical restriction to keep bond holdings away from zero for all dates. Recall that updating (5a), we have

$$(III.1) \quad \frac{M_{t+1}}{P_{t+1}} + k\left(\frac{L_{t+1}}{P_{t+1}} - \frac{M_{t+1}}{P_{t+1}}\right) = 1.$$

If the central bank's inflation target and initial interest rate happen to make P_{t+1} equal to L_{t+1} , the solution to (III.1) is $M_{t+1} = L_{t+1}$, and we get $B_{t+1} = 0$. To rule this out, note that from (8) and

$P_{t+1} = \Pi_t P_t$, it follows that $P_{t+1} = L_{t+1}$ only occurs at the critical value $P_{t+1}^c = L_t(1 - d_{t+1})^{-1}$,

which corresponds to $P_t^c = L_{t-1}[(1 - d_{t+1})\Pi_t - d_t]^{-1}$. P_t is a strictly monotone function of i_t in our model.¹ Therefore, we can rule out the critical value of P_t by ruling out (at most) a single (otherwise feasible) value of i_t .

Repeating this argument for dates $t+2$ and beyond, we get at most one more restriction on i_t to rule out the critical value of P_{t+j} for each j . So, given the central bank's choice of the inflation sequence, there is a countable set of initial values for i_t that the central bank cannot choose.

1. Given Π_t , an increase in i_t would raise b_t and reduce m_t by the same amount if P_t did not change; since $k'(b_t) < 1$, these changes would make $m_t + k(b_t) < 1$; so, P_t would have to fall to satisfy (5a).